Public Liquidity Demand and Central Bank Independence*

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Abstract

This paper studies how private demand for public liquidity affects the independence of a central bank vis-à-vis the fiscal authority. Whereas supplying liquidity to the private sector creates degrees of freedom for fiscal and monetary authorities vis-à-vis each other, we show that the authority that is most able to attract private liquidity demand can ultimately impose its views to the other.

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1 Introduction

A central bank is independent if the fiscal authority does not stand in the way of its objectives, foremost among them being price stability. In workhorse macroeconomic models, a necessary condition for equilibrium, the "intertemporal budget constraint of the government", imposes strong restrictions on jointly feasible fiscal and monetary policies. Sargent and Wallace (1981) famously deemed such a strong interdependence between fiscal and monetary policies an "unplesant arithmetic". In particular, the central bank is independent only if the fiscal authority can commit to a Ricardian policy, ensuring that the budget constraint holds for all paths of the price level.[1]

This paper extends these workhorse models in two directions. We first posit that the public sector has a unique ability to supply liquidity vehicles to the economy, thereby generating resources above and beyond fiscal surpluses. This may relax the interdependence between fiscal and monetary policies that derives from a standard intertemporal budget constraint. Second, rather than assuming that fiscal and monetary authorities indefinitely commit to policy rules, we endow both authorities with objectives and instruments, and study the subgame-perfect outcome from their strategic interactions. Put simply, we offer a formal game-theoretic analysis of Wallace's "game of chicken" between fiscal and monetary authorities.

One purpose of these extensions is to assess central-bank independence in the current context. Several observers (e.g., Blanchard, 2019) argue that the US government should reap the benefits from interest rates below growth rates by issuing more debt at zero fiscal and inflationary costs. Current attempts of the US executive branch at influencing monetary policy are apparently not perceived by markets as pure noise (Bianchi et al., 2019). Overall, relative to a view of the world in which central-bank independence is warranted by a Ricardian fiscal policy given the intertemporal budget constraint of the government, the current context suggests on the one hand that this budget constraint may be "soft", but that fiscal policy, on the other hand, is far from Ricardian. What is the net implication for central-bank independence?

Our analysis generates the following insights. First, we offer a general formulation of the condition under which public liquidity supply makes the monetary arithmetic "pleasant", in the sense that it relaxes fiscal and monetary interdependence and thus significantly expands the set of jointly feasible policies.[2] The condition is that the public sector must be able to indefinitely rollover securities that are not backed by any fiscal

[1] See Cukierman (2008) for a discussion of the legal and institutional determinants of central-bank independence.

[2] In response to Sargent and Wallace (1981), Darby (1984) offers an example of such a "pleasant monetarist arithmetic" and coins this term.

surplus, and that it must be able to do so at a sufficiently low cost relative to the return on pure consumption claims. A simple but noteworthy insight is that it suffices that only one type of public liabilities, e.g., central-bank reserves, can be rolled over at such a low cost for the monetary arithmetic to be pleasant even if other liabilities, such as government bonds, carry a higher yield. Under such pleasant monetary arithmetic, several price levels are consistent with a given path of fiscal surpluses.

One could conclude from this latter remark that demand for public liquidity reinforces central-bank independence. The predictions from our strategic analysis are however gloomier. We find that the degree of "pleasantness" of the economy, broadly defined as the wiggle room between fiscal and monetary policies, is actually not the essential determinant of central-bank independence when both authorities are strategic. It is rather the ability of the central bank to mop up private liquidity before the fiscal authority does so with the issuance of debt that warrants its independence. We find indeed that the authority that is the fastest at meeting private liquidity demand can force the other to chicken out. There is fiscal consolidation and a stable price level if the monetary authority preempts liquidity demand whereas there is fiscal expansion and inflation in case the fiscal authority does so.

Our explicit strategic approach offers useful insights into the question that Sargent and Wallace (1981) raise in conclusion of their unpleasant arithmetic: "The question is, Which authority moves first, the monetary authority or the fiscal authority? In other words, Who imposes discipline on whom?" We contend that the authority who moves first in this sense is the one that preempts private demand for liquidity. If the fiscal authority is the primary issuer of the public liabilities sold to the private sector, it de facto controls future price levels. If the monetary authority by contrast is this primary issuer, then fiscal paths cannot dictate the price level.

Despite the current prevalence of low interest rates relative to growth, and even if this prevalence implies that issuing government debt comes at no fiscal cost, our analysis therefore implies that capping the ratio of outstanding government debt to GDP remains a requirement for central bank independence and price stability. The quantitative value of such a cap depends on the intensity of the private demand for public liquidity relative to growth. We further discuss these policy implications in Section 3.4 and how they extend to, for example, a situation where inflation is below the central bank target.

The paper is organized as follows. For expositional clarity, we derive our main insights in a very simple overlapping-generations model in which informational asymmetries in the credit market create room for valuable public liquidity supply. Section 2 derives the implications of a pleasant monetary arithmetic in this simple framework, and Section 3 solves for Wallace's game of chicken in it. Section 4 is more abstract in nature. It offers a general and yet compact condition under which monetary arithmetic is pleasant in a broad class of models. It aims in particular at confirming that our main results do not live or die on the overlapping-generations structure used to illustrate them.

Related literature. This paper connects to the literature on the fiscal theory of the price level starting with Leeper (1991), Sims (1994), Woodford (1994, 1995, 2001), or Cochrane (2001, 2005), and to its criticism (see Buiter, 2002; McCallum, 2001; Niepelt, 2004, among others). It relates in particular to the literature investigating whether the fiscal theory of the price level applies in non-Ricardian environments, starting with Bénassy (2008). More recent contributions include Bassetto and Cui (2018) and Farmer and Zabczyk (2019). Our characterization of a pleasant monetary arithmetic builds on Bassetto and Cui (2018), who show that low interest rates on public debt prevents fiscal policy from selecting a unique price level. The simple economy in which we cast our game of chicken closely relates in particular to one of the models in Bassetto and Sargent (2019), in which public liabilities serve as liquidity vehicles and fund transfers that mitigate credit-market failures. Our main contribution relative to these papers is to go beyond the determination of the set of feasible fiscal and monetary policies and predict the ones that actually arise given the strategic interactions between fiscal and monetary authorities.

Our paper is also connected to an older literature (Alesina, 1987; Alesina and Tabellini, 1987; Tabellini, 1986, e.g.) that investigates the equilibria of games between multiple branches of government. More recent contributions include Dixit and Lambertini (2003) or Aguiar et al. (2015). The latter study fiscal and monetary policy in a monetary union with atomistic sovereigns that may default. Closer to our paper, Martin (2015) finds as we do that fiscal irresponsibility leads to long-term inflation even when the fiscal authority is not first mover in the sense of Sargent and Wallace (1981). Our contribution with respect to this literature is to model the game of chicken between monetary and fiscal authorities when both components of the public sector are strategic, and to allow for the possibility that both authorities supply securities with money-like properties to the private sector. This leads to the novel insight that the authority that preempts private liquidity demand imposes its views, no matter how "pleasant" the monetary arithmetic.

We also relate to the literature on rational bubbles in two ways. First, informational asymmetries in credit markets create room for bubbles in our overlapping-generations example, as they do in Farhi and Tirole (2012) or Martin and Ventura (2012). Second, and more importantly, we also relate to the literature linking monetary policy to bubbles, including Gali (2014). In particular, Asriyan et al. (2019) consider a competition between private bubbles and a public one ("money"). By contrast, we study competition between distinct public bubbles, reserves and government bonds, to study fiscal-monetary interactions.

The idea that public debt satisfies private liquidity demand goes back to at least Diamond (1965) and has been widely studied since (see Woodford, 1990; Holmström and Tirole, 1998, among others). Krishnamurthy and Vissing-Jorgensen (2012) showed in the data that public debt shared many of the properties of money.

Finally, this paper also connects to the literature that relates the independence of a central bank to the structure of its balance sheet (Sims, 2003; Hall and Reis, 2015). Whereas this literature argues that the need for recapitalization by the government threatens independence, we predict that large transfers to the government from the monetary authority are the smoking gun for loss of independence.

2 Pleasant monetary arithmetic: A simple example

This section uses a simple overlapping-generations framework to showcase how private demand for public liquidity renders the monetary arithmetic pleasant, in the sense that it relaxes the interdependence between monetary and fiscal policies.[3]

2.1 Setup

We write down a model in which the public sector may generate resources by satisfying private liquidity demand. Time is discrete and indexed by $t \in \mathbb{N}$. There is a single consumption good. The economy is populated by a public sector and by two types of private agents, savers and entrepreneurs. At each date, a unit mass of entrepreneurs and a unit mass of savers are born. They live for two dates and value consumption only when old, at which time they are risk-neutral. All agents use the same currency as a unit of account ("cashless economy").

Savers. Young date-t savers receive a real endowment that they can store with a linear return $e^{-\delta}$. Endowments are i.i.d. across savers of a given cohort. The date-t endowments' distribution has a unit mean and its support has a lower bound $\bar{\tau}_t \geq 0$. For simplicity, suppose that $\bar{\tau}_1 \in (0, 1)$, and that $\bar{\tau}_t = 0$ at every date $t \neq 1$.

Entrepreneurs. Young date-t entrepreneurs are endowed with a storage technology with a random linear return. The (gross) return has expected value e^{ρ}, where $\rho > \max\{-\delta; 0\}$, and its distribution has 0 in its support. Returns are perfectly correlated

[3]Section 4 presents a much more general formulation of the insights in this Section 2.

across entrepreneurs of the same cohort. Entrepreneurs are competitive in the credit market.

Public sector. The public sector sets at each date t the price level P_t and operates transfers to the private sector.

Remark on price-level determination. We deliberately posit that the public sector can implement whichever price level it wants without explicitly modelling a particular implementation. We believe that this is the right benchmark when studying how fiscal policy restricts the set of feasible monetary policies. Accordingly, the only restrictions on feasible price-level paths originate from fiscal policy in this paper. Studying how these restrictions interplay with specific implementations (e.g., interest rules) is an interesting route for future research.

Transfers. We denote by σ_t the date-t (real) transfer from young entrepreneurs to the public sector and by τ_t that from young savers to the public sector. We omit transfers involving old agents to save on notations and because they will play no role in the subsequent strategic analysis in Section 3. The analysis is verbatim if we include them, though. The date-t real fiscal surplus, s_t, is thus simply

$$s_t = \sigma_t + \tau_t. \tag{1}$$

The public sector can issue one-period nominal bonds. Let $D_t \geq 0$ denote the number of currency units due at date $t+1$ and promised at date t.[4] The date-t real price of public debt is $\phi_t \geq 0$.

Finally, the public sector starts out with an exogenous legacy nominal liability $L > 0$ due at date 1. It can repay it at date 1, but can also buy back all or part of it at the outset at date 0 at the prevailing price ϕ_0. The existence of such a "long-term" (i.e., due after date 0) legacy liability is an important ingredient in the game of chicken studied in Section 3. We introduce this ingredient in the simplest possible way, but our insights would carry over with legacy liabilities due at multiple dates (including at $t = 0$).[5] We denote by $L_{PS} \in [0, L]$ the nominal amount of the date-1 legacy liability bought back at date 0.

One possible interpretation of this legacy liability is that the public sector has issued long-term debt in an unmodelled past (before date 0) and that L is the residual amount

[4] The exclusion of public savings could be replaced by a no-Ponzi game condition without adding new insights.

[5] Appendix A discusses such an extension in which there are legacy liabilities due at several dates.

due at date 1. An alternative interpretation is that of a bailout decision following a major financial crisis. Under this interpretation, the liabilities L are that of a distressed (unmodelled) financial sector, and "default" would therefore correspond to an incomplete bailout.

Information structure. The public sector does not observe savers' endowments, entrepreneurs' realized returns, nor trades by private agents. There exists $T \in \mathbb{N}$ such that if t does not belong to $\{1+k(T+1), k \in \mathbb{N}\}$, then savers born at date t perfectly observe the return realized by date-t entrepreneurs at date $t+1$. Otherwise, they do not observe it.

That $\rho > -\delta$ implies that (risky) loans from savers to entrepreneurs unlock gains from trades. Such a private credit market works seamlessly for the cohorts that do not experience any informational asymmetries between lenders and borrowers. At dates that belong to $\{1 + k(T + 1), k \in \mathbb{N}\}$, however, the credit market collapses as entrepreneurs can always claim at the next date that their realized return is zero. Thus they cannot pledge any future output to savers. Accordingly, we interpret the dates at which there are no informational asymmetries between savers and entrepreneurs as "normal times," and the ones in which the credit market shuts down as "financial crises".[6]

2.2 Feasible policies

A feasible policy consists in $L_{PS} \in [0, L]$ and in a sequence $(D_t, \sigma_t, \tau_t, P_t)_{t\in\mathbb{N}}$ such that (i) the public sector is solvent; (ii) markets clear; (iii) private agents optimize; (iv) savers are indifferent between public and private investment.[7] This latter condition (iv) implies that $\phi_t = e^{\delta}$ when t belongs to $\{1 + k(T + 1), k \in \mathbb{N}\}$, and $\phi_t = e^{-\rho}$ otherwise. We stay short of fully characterizing the set of feasible policies here, and merely study how the private demand for liquidity affects the size of this set of feasible policies.

At every date $t \in \mathbb{N}$, the budget constraint of the public sector reads

$$\mathbb{1}_{\{t=1\}}\frac{L - L_{PS}}{P_t} + \frac{D_{t-1}}{P_t} = \sigma_t + \tau_t + \phi_t\frac{D_t - \mathbb{1}_{\{t=0\}}L_{PS}}{P_{t+1}}, \tag{2}$$

where $D_{-1} = 0$ by convention.

Notice that the public sector cannot raise taxes other than from young savers at date 1. Given the distribution of endowments and returns, savers and entrepreneurs can indeed always claim that they are penniless so as to avoid taxation, except for young

[6] Note that the case $T = 0$ is essentially a situation of dynamic inefficiency à la Wallace (1980).

[7] Condition (iv) is only meant to eliminate corner equilibria due to a linear model. It would be unnecessary if, e.g., savers' storage technology had strictly decreasing returns.

date-1 savers who all own at least $\bar{\tau}_1$.[8] This entails that a feasible policy must be such that $\sigma_t \leq 0$ for all $t \in \mathbb{N}$, $\tau_t \leq 0$ for all $t \neq 1$, and $\tau_1 \leq \bar{\tau}_1$. This implies in turn that for all $t > 1$,

$$\phi_t \frac{D_t}{P_{t+1}} \geq \frac{D_{t-1}}{P_t}. \tag{3}$$

Unpleasant monetary arithmetic. Suppose first that financial crises are not frequent: $T > \delta/\rho$. As a result, $e^{-\delta}\left(e^{\rho}\right)^T > 1$, and so the public sector cannot issue debt after date 0 because (3) implies that any amount to be refinanced would ultimately exceed savers' aggregate endowment. Hence, feasible policies are such that $D_t = 0$ for all $t \in \mathbb{N}\backslash\{0\}$, and so $\sigma_t = \tau_t = 0$ for all $t > 1$. The date-0 and date-1 budget constraints imply:

$$\frac{L}{P_1} = \frac{\sigma_0 + \tau_0}{\phi_0} + \sigma_1 + \tau_1. \tag{4}$$

In sum, the feasible fiscal policies are such that $\sigma_t = \tau_t = 0$ for all $t > 1$ and are thus given by the quadruplets $(\sigma_0, \sigma_1, \tau_0, \tau_1) \in (-\infty, 0]^3 \times (-\infty, \bar{\tau}_1]$ such that $s_0/\phi_0 + s_1 > 0$.

Of particular interest to us is the fact that every such feasible fiscal policy is associated with a unique feasible initial price level P_1 determined by (4). In this case, we deem the monetary arithmetic *unpleasant*, in the sense that any reduction in fiscal surpluses must come with a higher price level P_1 (see Section 4 for a more formal definition).

Pleasant monetary arithmetic. Suppose now that $T \leq \delta/\rho$. Consider $(\sigma_0, \sigma_1, \tau_0, \tau_1)$ a fiscal policy that is feasible when the monetary arithmetic is unpleasant. *By contrast, when $T \leq \delta/\rho$, it is no longer the case that this fiscal policy uniquely pins down the feasible date-1 price P_1.* For every $\omega \in [0, 1 - \tau_1]$, the date-1 price level $P_1(\omega)$ defined as

$$\frac{L}{P_1(\omega)} = \frac{\sigma_0 + \tau_0}{\phi_0} + \sigma_1 + \tau_1 + \omega \tag{5}$$

is indeed feasible. This is simply because $\left(e^{\rho}\right)^T e^{-\delta} \leq 1$ implies that the "bubble" ω can be rolled over after date 1, as it will never exceeds disposable savings.

In this case, the monetary arithmetic is *pleasant*, in the sense that reductions in fiscal surpluses need not imply an increase in P_1 as they may correspond to an increase in the proceeds ω from supplying liquidity to the private sector.

To wrap up, if $T > \delta/\rho$, then fiscal and monetary policies are strongly interdependent

[8] Note that we rule out multilateral surplus extraction mechanisms that could at least partially elicit entrepreneurs' realized returns.

in the sense that each feasible fiscal policy is associated with a unique price level P_1. As in the fiscal theory of the price level, real fiscal surpluses dictate the price level. In particular, if a feasible policy features lower surpluses than another one, then it must also come at a higher price level. If more frequent crises imply a higher demand for public liquidity ($T \leq \delta/\rho$), fiscal and monetary policies are less interdependent in the sense that a given feasible fiscal policy is associated with an interval of date-1 price levels. Two feasible policies can be such that one of them features both lower surpluses and a lower date-1 price level than the other.

2.3 Feasible policies with multiple public liabilities

In order to prepare the ground for the game of chicken studied in Section 3, this subsection briefly revisits the above analysis of feasible policies in the case in which the public sector is split into fiscal and monetary authorities that have separate budget constraint, as in Bassetto and Messer (2013) or Hall and Reis (2015).

Suppose thus that the public sector is comprised of two authorities, a fiscal one F and a monetary one M. At each date t, the fiscal authority F sets the transfers σ_t and τ_t and issues government bonds. The monetary authority sets the price level P_t and issues remunerated reserves. We respectively denote by $B_t \geq 0$ and $X_t \geq 0$ the number of date-$(t+1)$ currency units respectively promised by F and M at date t, and by L_F and L_M their respective date-0 buybacks of the exogenous liability L. Both authorities can also transfer resources to each other. We denote θ_t the date-t transfer from the monetary authority to the fiscal one. This transfer can be either positive (dividend) or negative (recapitalisation).

Budget constraints. The budget constraints of M at dates 0 and 1 read

$$0 = -\theta_0 + \phi_0 \left(\frac{X_0}{P_1} - \frac{L_M}{P_1} \right), \tag{6}$$

$$\frac{X_0}{P_1} - \frac{L_M}{P_1} = -\theta_1 + \phi_1 \frac{X_1}{P_2}, \tag{7}$$

whereas that of F are

$$0 = \theta_0 + \sigma_0 + \tau_0 + \phi_0 \left(\frac{B_0}{P_1} - \frac{L_F}{P_1} \right), \tag{8}$$

$$\frac{B_0}{P_1} + \frac{L}{P_1} - \frac{L_F}{P_1} = \theta_1 + \tau_1 + \sigma_1 + \phi_1 \frac{B_1}{P_2}. \tag{9}$$

This leads to a consolidated intertemporal budget constraint similar to (5):

$$\frac{L}{P_1} = \frac{\sigma_0 + \tau_0}{\phi_0} + \sigma_1 + \tau_1 + \phi_1 \left(\frac{B_1}{P_2} + \frac{X_1}{P_2} \right). \tag{10}$$

Note that fiscal and monetary liabilities perform the exact same economic role as liquidity storages. Section 3 briefly discusses the case in which the monetary ones also serve as means of payment.

If $T > \delta/\rho$, then neither F nor M can issue liabilities after date 0 nor operate transfers after date 1, and the fiscal policy $(\sigma_0, \sigma_1, \tau_0, \tau_1)$ pins down a unique date-1 price level P_1. When the arithmetic is pleasant, conversely, for a given feasible fiscal policy $(\sigma_0, \sigma_1, \tau_0, \tau_1)$ and a given date-1 real government debt (unbacked by future real surpluses) $b_1 = \phi_1 B_1/P_2 < 1 - \tau_1$, there exists a continuum of feasible date-1 real values of central-bank reserves $x_1 \in [0, 1 - b_1 - \tau_1]$. To each level of x_1 corresponds a unique price level P_1 given by:

$$\frac{L}{P_1} = \frac{\sigma_0 + \tau_0}{\phi_0} + \sigma_1 + \tau_1 + b_1 + x_1. \tag{11}$$

In sum, when the arithmetic is pleasant, there exists an interval of feasible price levels given both fiscal surpluses and the issuance patterns of government bonds. The larger the bubble that accrues to the fiscal authority, the less the monetary authority has wiggle room, however.

With multiple maturities. Appendix A extends these results to the situation in which legacy debt is due at multiple maturities. In such a context, Cochrane (2001) shows that the issuance policy of public liabilities at date 0 matters for the exact price level path. We extend this result and show that when the central bank can issue remunerated reserves, the central bank can pin down the price levels at dates 0 and 1.

We have focussed thus far on determining how the "pleasantness" of the monetary arithmetic shapes the set of feasible prices given fiscal policy. We have shown in particular that when both branches of government supply liquidity vehicles to the private sector, then not only fiscal surpluses, but also the real resources stemming from liquidity supply, contribute to pin down the price level. These additional resources expand the sets of feasible policies. It would be incorrect to infer that such an expansion automatically reinforces central-bank independence. Only a model that compares strategic interactions between fiscal and monetary authorities under varying degrees of "pleasantness" can guide an analysis of the impact of public liquidity supply on central-bank independence. In short, independence must be assessed in a model in which it matters. The following section offers such a model.

3 Wallace's game of chicken

This section solves an explicit model of Wallace's "game of chicken" in the economy outlined in the previous section. In this game, the fiscal authority (F) and the monetary authority (M) have to repay the liability L that is due at date 1,[9] but they disagree on objectives: the monetary authority cares about price stability and the fiscal authority values transfers to entrepreneurs.

To set up a game between F and M, we first specify their respective objectives and describe the timing of their interactions. We then determine the equilibrium outcome of the game under both unpleasant and pleasant arithmetic. We finally derive policy implications and consider possible extensions of the game.

Objectives of F and M. The respective date-t objectives of F and M are:

$$U_t^F = -\sum_{t' \geq 0} \beta^{t'} (\sigma_{t+t'} + \alpha_F \Delta_{t+t'}), \tag{12}$$

$$U_t^M = -\sum_{t' \geq t} \beta^{t'} (\mid P_{t+t'} - P_M \mid + \alpha_M \Delta_{t+t'}), \tag{13}$$

where $\beta \leq e^{-\rho}$ and $\alpha_F, \alpha_M, P_M > 0$. The variable Δ_t is equal to 1 in case of an outright default of the public sector on any of its claims held by the private sector due at date t, and to 0 otherwise.

In words, each authority $X \in \{F; M\}$ incurs a cost α_X if the public sector nominally defaults. The fiscal authority also values subsidies to young entrepreneurs (but does not care about the price level), whereas the monetary authority also finds it costly to deviate from a given target P_M for the price level (but does not care about transfers). To lift equilibrium indeterminacy, we will also posit that M, when indifferent among several actions, prefers the ones that maximize transfers to young entrepreneurs.

Where do these objectives come from? For brevity, we simply posit that the public sector is comprised of two distinct authorities with different objective functions. Yet a simple time-inconsistency argument could micro-found the delegation of price-level determination to a biased monetary authority. Suppose that the social welfare function puts more weight on entrepreneurs than on savers, but that the government lacks commitment. In this case, if private agents use nominal contracts, such a government would be

[9] If the liability is only due at date 0, the game would be a one-shot static game without any role for the issuance of public liabilities.

tempted to inflate away old entrepreneurs' debts ex-post so as to transfer consumption to them from old savers. Savers would anticipate this, and this would inefficiently shut down credit markets ex-ante. Delegation to an entity with a mandate for a stable price level solves this problem. Our setting is one in which this entity cannot perfectly commit to a path of price levels, however, because it also cares about default. Another way of saying this is that the case in which $\alpha_M = 0$ is that in which M can fully and credibly commit to set the price level to P_M, as setting $P_t = P_M$ at all dates clearly is a dominant strategy in this case.

Actions and timing. At each date $t \geq 0$,

- M first sets the date-t price level P_t.
- F taxes young savers an amount τ_t up to $\bar{\tau}_t$.
- Young savers issue (nominal) demands $\bar{B}_t$ and $\bar{X}_t$ for claims issued by F and M respectively at the prevailing market price ϕ_t, where $\bar{B}_t, \bar{X}_t \geq 0$.[10] We adopt the convention that $\bar{B}_{-1} = \bar{X}_{-1} = 0$.
- F decides on a supply $B_t \in [0, \bar{B}_t]$ and M on a supply $X_t \in [0, \bar{X}_t]$ and they collect their respective proceeds $\phi_t B_t / P_{t+1}$ and $\phi_t X_t / P_{t+1}$. If one authority leaves a residual demand unsatisfied, then the other is entitled to satisfy it.
- One authority makes a take-it-or-leave-it offer to the other that consists in a real transfer $-\sigma_t$ to young date-t entrepreneurs and a reimbursement to the holders of current liabilities. Current liabilities are the endogenous ones B_{t-1} and X_{t-1} at all dates and the exogenous one L at date 1. At date-0, the offer also includes the full or partial buyback of the exogenous liability L, either by the central bank (L_M) or by the fiscal authority (L_F). If the receiving authority turns down the offer then each authority uses its proceeds as it sees fit.

3.1 Equilibrium

Our equilibrium concept imposes the natural requirements that the sequences of private demands for public liabilities $\bar{B}_t$ and $\bar{X}_t$ solve the savers' problem given the policies decided by fiscal and monetary authorities, and that these authorities play a subgame-perfect Nash equilibrium given this private liquidity demand. More precisely:

[10] Whereas such demands are expressed in nominal or real terms is immaterial in our flexible price, perfect-foresight environment.

Definition 1. ***(Equilibrium)*** *An equilibrium is a policy sequence $(B_t, X_t, P_t, \sigma_t, \tau_t)_{t\geq 0}$, date-0 buybacks (L_F, L_M), and a sequence of liquidity demand $(\bar{B}_t, \bar{X}_t)_{t\geq 0}$ such that all agents have perfect foresight and for all $t \geq 0$,*

- *Savers are indifferent between private and public investments and their demands for the latter are rational given prices and the policy sequence:*

$$\phi_t \frac{\bar{X}_t + \bar{B}_t}{P_{t+1}} \leq 1, \tag{14}$$

$$\mathbb{1}_{\{t=1\}} \frac{L - L_F - L_M}{P_t} + \frac{\bar{X}_{t-1} + \bar{B}_{t-1}}{P_t} \leq -\mathbb{1}_{\{t=0\}} \phi_t \frac{L_F + L_M}{P_{t+1}} + \phi_t \frac{X_t + B_t}{P_{t+1}} + \sigma_t + \tau_t, \tag{15}$$

$$\phi_t = e^{\delta} \text{ if } t \in \{1 + k(T+1), k \geq 0\}, \;\; \phi_t = e^{-\rho} \text{ otherwise.} \tag{16}$$

- *Given demands $(\bar{B}_t, \bar{X}_t)_{t\geq 0}$ and prices $(\phi_t)_{t\geq 0}$, the date-t continuation of the policy sequence constitutes a subgame perfect Nash equilibrium between F and M and satisfies:*

$$\sigma_t \leq 0 \text{ and } \tau_t \leq \bar{\tau}_t, \tag{17}$$

$$B_t \leq \bar{B}_t \text{ and } X_t \leq \bar{X}_t, \tag{18}$$

$$L_F + L_M \leq L. \tag{19}$$

Condition (14) ensures that savers' total unit endowment suffices to fund their real demands for the public sector's liabilities. Condition (15) imposes that savers rationally anticipate default along the equilibrium path. Inequality (17) reflects that entrepreneurs cannot be taxed and young savers can be taxed only if the lower bound of their endowment is strictly positive. Condition (18) ensures that demand for public liabilities weakly exceeds supply. Finally, condition (19) limits the quantity of buybacks at date 0.

In sum, an equilibrium is such that i) the flows between private and public sectors are feasible, ii) F and M play a subgame-perfect Nash equilibrium given savers' demands for public liabilities, and iii) savers invest optimally, rationally anticipating this game.

When the monetary arithmetic is pleasant, the public sector can earn resources by issuing unbacked reserves and bonds—"bubbles". There are of course many feasible paths for such bubbles, including a non-bubbly one, and we do not want to arbitrarily select a particular pattern, nor do we want to arbitrarily enable the public sector to pick one. Our formalization of the issuance process, whereby savers submit demands for liquidity vehicles, and our equilibrium requirement that these demands and the supply of the public sector are merely sustainable, accordingly ensures that we do not arbitrarily rule out any possible (deterministic) pattern of bonds and reserves issuance.

After date 1, in the absence of any tax revenue, the net aggregate real flow received from savers by the public sector at date t, that we denote by λ_t,

$$\lambda_t = \phi_t \frac{X_t + B_t}{P_{t+1}} - \frac{X_{t-1} + B_{t-1}}{P_t}, \tag{20}$$

is a bubble starting at date t. Strictly positive bubbles clearly exist if and only if the arithmetic is pleasant ($T \leq \delta/\rho$).

The rest of the section characterizes such equilibria, considering in turn the cases of unpleasant and pleasant monetary arithmetics. In order to ease exposition, we study the limiting case in which both authorities prefer any outcome to sovereign default:

$$\alpha_F = \alpha_M = +\infty. \tag{21}$$

Section 3.5.1 shows that our insights carry over with finite default costs.

3.2 Unpleasant arithmetic

Suppose first that $T > \delta/\rho$. We have seen that all public liabilities must be backed by fiscal resources in this case. Given that there are no tax revenues at other dates than 1, any equilibrium must be such that for all $t \geq 1$,

$$\bar{X}_t = \bar{B}_t = 0, \tag{22}$$

and the equilibria are as follows.

Proposition 1. ***(Game of chicken under unpleasant arithmetic)*** *Let (b, x) two positive numbers such that $b+x \leq \bar{\tau}_1$ and $(b, x) \neq (\bar{\tau}_1, 0)$. There exists a unique equilibrium associated with (b, x). It is such that $\bar{B}_0 = B_0 = bP_1$ and $\bar{X}_0 = X_0 = xP_1$. This equilibrium is default-free and has the following characteristics:*

- *At date 0, M sets $P_0 = P_M$ and uses its resources $\phi_0 x$ to buy back all or part of L. F uses its resources $\phi_0 b$ to subsidize young entrepreneurs.*
- *At date 1, M sets $P_1 = \max\{P_M; L/(\bar{\tau}_1 - b)\}$. F exhausts its fiscal capacity $\bar{\tau}_1$ and uses it to pay back b, x, $(L/P_1 - x)^+$, and to subsidize young entrepreneurs if possible. There are no residual fiscal resources available for transfers to young entrepreneurs if $P_1 > P_M$.*

- *At all dates $t \geq 2$, the price level is P_M and there are no transfers between the public and the private sector.*

The other equilibria are such that $\bar{B}_0 = B_0 = \bar{\tau}_1 P_1$ and $\bar{X}_0 = X_0 = 0$. In this case, F must buy back the entire liability L at date 0, and P_1 can reach any level above $\max\{P_M; L/\bar{\tau}_1\}$.

Proof. See Appendix B.1. □

These equilibrium actions admit the following practical intepretations. At date 0, M enters into open-market operations. It issues reserves to purchase as much as possible of the outstanding public liability L and possibly pay a dividend to F in case of full buyback of L. F uses the proceeds from bond issuance and this dividend to subsidize young entrepreneurs. At date 1, the taxation of young savers serves to pay back L, B_0, and to recapitalize the central bank so that it can refinance the liability it issued at date 0.

To gain intuition for these results, let us clarify how the price level depends on the issuance policies of both monetary and fiscal authorities. At date-1, the budget constraints of the public sector require the price level to satisfy:

$$P_1 = \frac{L + (D_0 - L_{PS})}{\sigma_1 + \tau_1}. \tag{23}$$

As a result, for a given stream of fiscal surpluses, the key determinant of the price pattern is the date-0 debt issuance that is not used to buyback L, that is, $D_0 - L_{PS} = B_0 - L_F + X_0 - L_M$, that is jointly determined by fiscal and monetary authorities.

The fiscal authority seeks to induce the largest possible date-1 price level so as to reduce the real value of L and thus devote the maximum amount of fiscal resources to subsidizing young entrepreneurs. The monetary authority by contrast prioritizes price stability. Accordingly, F seeks to maximize $B_0 - L_F$ by setting $B_0 = \bar{B}_0$ and $L_F = 0$. M seeks to preempt liquidity from F by setting $X_0 = \bar{X}_0$ and minimizes $X_0 - L_M$ by setting $L_M = X_0$.

The case in which $(b, x) = (\bar{\tau}_1, 0)$ is degenerate. When F can borrow against its entire date-1 fiscal capacity, it must buy back L entirely to avoid default. The real cost of this buyback depends on savers' (self-justified) anticipation of the date-1 price level.

Strategic fiscal irresponsibility. In sum, F can make a strategic use of fiscal irresponsibility that forces M to accommodate at lower debt levels L than $\bar{\tau}_1 P_M$, the fiscal capacity of the public sector at the target price level P_M. By spending its entire initial resources $\phi_0 b$ on young entrepreneurs rather than on reimbursing L, F ensures that the date-1 outstanding debt is sufficiently large relative to the date-1 resources of the public sector that M, despite moving first at the outset of date 1, has no other option but accommodating with $P_1 > P_M$ as soon as $L > (\bar{\tau}_1 - b)P_M$, which is smaller than date-1 fiscal capacity $\bar{\tau}_1 P_M$. Such strategic fiscal irresponsibility is all the more effective because b is a large fraction of $\bar{\tau}_1$.

The authority that preempts liquidity imposes its views. One can formalize this as follows. From Proposition 1, the set of equilibria such that $\bar{B}_0 < \bar{\tau}_1 P_1$ can be indexed by the set of positive numbers (b, x) such that $b + x \leq \bar{\tau}_1$ and $(b, x) \neq (\bar{\tau}_1, 0)$. Denote $\mathcal{E}$ this set of equilibria. The equilibrium associated with (b, x) is such that $\phi_0 b$ (respectively $\phi_0 x$) are the real resources collected by F (M respectively) at date 0.

Proposition 2. ***(The authority that preempts liquidity imposes its views.)*** *Over $\mathcal{E}$, the utility of M (given by (12)) is (weakly) decreasing in b whereas that of F (given by (13)) is (weakly) increasing in b.*

- ***Corollary 1**: Over equilibria in $\mathcal{E}$ associated with the same level of date-0 total public resources $\phi_0(b + x)$, each authority's utility increases in its share in these total resources.*

- ***Corollary 2**: Suppose that two equilibria in $\mathcal{E}$ are associated with distinct levels of date-0 total public resources $\phi_0(b + x)$. M may well be better off in the one with the smallest total resources if it attracts a larger fraction of them at date 0.*

Proof. In any equilibrium within $\mathcal{E}$, M sets the price at P_M at all dates but 1 at which $P_1 = \max\{P_M; L/(\bar{\tau}_1 - b)\}$. F can spend $\phi_0 b$ at date 0 and, viewed from date 0, $\max(0, \beta(\bar{\tau}_1 - b - L/P_1))$ at date 1. This readily implies the claimed variations with respect to b for their respective preferences. □

The utility of M really is *an index of central-bank independence as we defined it in the introduction* over the equilibria in $\mathcal{E}$ because it is only affected by, and decreasing in, date-1 departures from the target price level P_M. Proposition 2 substantiates our claim that the independence of the central bank depends on its ability to preempt liquidity. Corollary 1 states that the central bank is more independent if it attracts a larger fraction of total date-0 liquidity. Corollary 2 states that the central bank may actually end up more constrained by the fiscal authority even if the public sector has overall more date-0 resources if the fiscal authority ends up taking over these resources.

Proposition 2 focuses on equilibria in which $\bar{B}_0 < \bar{\tau}_1 P_1$ because the case in which F can borrow against its entire date-1 fiscal capacity at date 0 is degenerate. All the date-1 price levels above a given threshold can be imposed on M at date 1 because they correspond to self-justified date-0 savers' beliefs. If one selects in this case the outcome $P_1 = +\infty$, then one can expand the set of equilibria to which Proposition 2 applies to these degenerate equilibria.

3.3 Pleasant arithmetic

We now characterize equilibria when the monetary arithmetic is pleasant ($T \leq \delta/\rho$). We mainly aim at showing that both insights above—i) F is strategically fiscally irresponsible, and ii) the authority that preempts liquidity imposes its views—hold in exactly the same way as when the arithmetic is unpleasant. The only difference with the unpleasant case is that the public sector now has the ability to collect additional resources from the issuance of unbacked securities. How F and M strategically use these resources remains however unchanged.

For expositional simplicity only, we assume away any fiscal resources by setting $\bar{\tau}_1 = 0$. In addition, we consider only equilibria such that there is a demand for reserves at each date: For all $t \geq 0$, $\bar{X}_t > 0$. We deem such equilibria "liquid".[11] Section 3.5.2 shows that our insights still hold over all possible equilibria.

Proposition 3. ***(Game of chicken under pleasant arithmetic)*** *Let $\Lambda = (b_t, x_t)_{t \geq -1}$ a sequence of positive numbers such that $b_{-1} = x_{-1} = 0$ and for all $t \geq 0$,*

$$x_t > 0, \tag{24}$$

$$b_{t-1} + x_{t-1} \leq \phi_t(b_t + x_t) \leq 1. \tag{25}$$

There exists a unique liquid equilibrium associated with Λ. It is such that $\bar{B}_t = B_t = b_t P_{t+1}$ and $\bar{X}_t = X_t = x_t P_{t+1}$. Reciprocally, to every liquid equilibrium corresponds a sequence Λ that satisfies (25). Every liquid equilibrium is default-free and has the following characteristics:

- *At date 0, M sets $P_0 = P_M$ and uses its resources $\phi_0 x_0$ to buy back all or part of L. F uses its resources $\phi_0 b_0$ to subsidize young entrepreneurs.*
- *At date 1, M sets $P_1 = \max\{P_M; L/[\phi_1(b_1 + x_1) - b_0]\}$. The public sector uses $\phi_1(b_1 + x_1)$ to pay back b_0, x_0, $(L/P_1 - x_0)^+$, and to subsidize young entrepreneurs if possible. There are no residual fiscal resources available for transfers to young entrepreneurs if $P_1 > P_M$.*
- *At all dates $t \geq 2$, the price level is P_M and the net resources (20) are transferred to current young entrepreneurs.*

Proof. See Appendix B.2. □

The features of liquid equilibria at dates 0 and 1 are verbatim that in the case of unpleasant monetary arithmetic, up to the substitution of $\bar{\tau}_1$ with $\phi_1(b_1 + x_1)$. This

[11] Liquid equilibria are the counterpart under a pleasant arithmetic of the non-degenerate equilibria such that $(b, x) \neq (\bar{\tau}_1, 0)$ under an unpleasant one.

reflects that the public sector has no date-1 fiscal capacity (since we posited for simplicity $\bar{\tau}_1 = 0$), but that it collects resources $\phi_1(b_1 + x_1)$ from supplying liquidity. More generally, the public sector can issue unbacked securities ("bubbles") at possibly any date whereas it can only issue date-0 securities backed by date-1 taxes when the arithmetic is unpleasant.

The interactions between the monetary and the fiscal authorities can also be observed through the joint budget constraint as with equation (23)

$$P_1 = \frac{L + (D_0 - L_{PS})}{\sigma_1 + \tau_1 + \omega}. \tag{26}$$

where $\omega \in [0, 1 - \tau_1]$ when the monetary arithmetic is pleasant. Whereas the nature of the public-sector resources differ when the arithmetic is pleasant, their use does not: F is strategically fiscally irresponsible. As a result, the authority that preempts liquidity imposes its views.

Strategic fiscal irresponsibility. For a given Λ, if L/P_M exceeds the date-1 total public resources ($L \geq \phi_1(b_1 + x_1)P_M$), then the public sector has no choice but inflating away L at date 0 and setting $P_1 > P_M$. It is still the case that F can make a strategic use of fiscal irresponsibility that forces M to accommodate at lower debt levels than $\phi_1(b_1 + x_1)P_M$. By spending its entire initial resources $\phi_0 b_0$ on young entrepreneurs rather than on reimbursing L, F ensures that M has no other option but accommodating with $P_1 > P_M$ as soon as $L > [\phi_1(b_1 + x_1) - b_0]P_M$, which is smaller than $\phi_1(b_1 + x_1)P_M$. Such strategic fiscal irresponsibility is all the more effective because b_0 is a large fraction of $\phi_1(b_1 + x_1)$.

The authority that preempts liquidity imposes its views. One can formalize this as follows. From Proposition 3, the set of liquid equilibria can be indexed with the sequences Λ defined as in the proposition. Let $\mathcal{E}((b_t + x_t)_{t \geq 1})$ the set of liquid equilibria that share the same value of aggregate public liquidity supply $b_t + x_t$ from date 1 on.

Proposition 4. ***(The authority that preempts liquidity imposes its views.)*** *Over $\mathcal{E}((b_t + x_t)_{t \geq 1})$, the utility of M (given by (12)) is (weakly) decreasing in b_0 whereas that of F (given by (13)) is (weakly) increasing in b_0.*

- ***Corollary 1:*** *Over equilibria in $\mathcal{E}((b_t + x_t)_{t \geq 1})$ associated with the same level of date-0 total public resources $\phi_0(b_0 + x_0)$, each authority's utility increases in its share in the total resources.*

- ***Corollary 2:*** *Suppose that two equilibria in $\mathcal{E}((b_t + x_t)_{t \geq 1})$ are associated with distinct levels of date-0 total public resources $\phi_0(b_0 + x_0)$. M may well be better off* *in the one with the smallest total resources if it attracts a larger fraction of them at date 0.*

Proof. In any equilibrium, M sets the price at P_M at all dates but 1 at which $P_1 = \max\{P_M; L/[\phi_1(b_1 + x_1) - b_0]\}$. F can spend $\phi_0 b_0$ at date 0 and, viewed from date 0, $\beta \max(0, \phi_1(b_1 + x_1) - b_0 - L/P_1)$ at date 1. This implies the claimed variations with respect to b_0 for their respective preferences. □

The symmetry between Propositions 1 and 3 on one hand, and 2 and 4 on the other hand, showcases that a pleasant monetary arithmetic fuelled by a large demand for public liquidity need not reinforce central-bank independence, even though it expands the set of feasible fiscal and monetary policies. The main driver of central-bank's independence in our setup is its ability to preempt liquidity demand so that it does not get cornered into inflating debt away by a strategically irresponsible fiscal authority.

Accordingly, public liquidity demand would strengthen central-bank independence only if M was the dominant supplier of liquidity in the following sense: If, for some reason that is beyond the scope of our model, savers were coordinating only on equilibria such that b_0 is sufficiently small that fiscal irresponsibility never pays off. The current situation in many countries seems by contrast better described by equilibria in which both F and M are able to extract significant yields from their securities.

What if central-bank liabilities are directly inflationary? As already mentioned, one could assign other economic functions to the liabilities issued by the monetary authority than mere liquidity vehicles. If they served for example as exclusive means of payment, then a quantity equation could tie the price level to the quantity of outstanding liabilities. Whereas a full-fledged analysis is in order, we conjecture that this would considerably weaken central-bank independence. The monetary authority would indeed find itself between a rock and a hard place, or between current and future inflation. Either the central bank lets the fiscal authority preempt liquidity in order to force future monetary accommodation, or it preempts liquidity itself, thereby issuing liabilities that are inflationary right away.

3.4 Policy implications

The paper thus far has taken two steps to arrive at a model that help analyze central-bank independence in a context of "low rates" and non-Ricardian fiscal policy.

In the first step of Section 2, we endow the public sector with the ability to generate non-fiscal real resources by supplying liquidity vehicles to the private sector. These additional resources create wiggle room between fiscal and monetary policies, implying for example that the path of fiscal surpluses does not suffice to pin down the price level, even when the public sector issues only one-period liabilities.

In the second step of this section we explicitly model the reason an independent central bank is desirable in the first place—lack of commitment by the fiscal authority. We show that public liquidity supply may actually weaken central-bank independence if the fiscal authority is the primary issuer of liquidity vehicles.

Fiscal requirements are still needed when the monetary arithmetic is pleasant. A first policy implication is that fiscal requirements are still needed when the monetary arithmetic is pleasant: the fiscal authority cannot freely use the additional resources implied by low rates without jeopardizing central bank independence. This result applies to any type of government consumption, investment and subsidies, that do not relax the government's intertemporal budget constraint, that is, that are not associated with an increase in the net present value of future government ressources.

A possible fiscal requirement can consist in putting a cap on public-debt issuance even if the public sector can issue liabilities at a seemingly very low fiscal cost. Such a cap, if well designed, limits the capacity of the government to attract the private demand for public liquidity and should eventually ensure central bank independence. The impact of low rates on such rules should be merely quantitative, if any.

What if inflation is below target? A second policy implication comes from reinterpreting the insight that current spending by the fiscal authority makes it more difficult for the central bank to meet its objectives in the future. Suppose that for any reason, the central bank needs to commit to a date-1 price level that is above the one that it will find ex-post optimal (as with forward guidance). If it receives a large demand for reserves at date 0, then the central bank does not in this case buy back outstanding bonds with the proceeds as in our paper. It pays instead a special dividend to the fiscal authority who then spends it ("helicopter money"). This way, the central bank credibly commits to future ex-post excessive accommodation.

3.5 Extensions

This section extends the analysis to situations where the government can default—the central bank and the government have finite default costs, and to situations where the equilibrium can be illiquid—i.e., such that the demand for reserves can be 0. We also briefly discuss the application of our game-theoretic framework to a new Keynesian environment.

3.5.1 Sovereign default

Whereas the assumption of infinite costs of default simplified the exposition, the important insights carry over when these costs are finite. Suppose $\alpha_M, \alpha_F > 0$ in the setting described in subsection 3.3. For brevity and realism, we restrict the analysis to the case in which $\alpha_F > 1$. As will be clear below, this implies that M is always the authority that pulls the trigger on default by refusing arbitrarily large levels of inflation. The liquid equilibria have the very same structure as in the limiting case of infinite default costs, the only difference being that strategic fiscal irresponsibility is less effective at leading M to chicken out.

Proposition 5. ***(Liquid equilibria with finite default costs)*** *There is a one-to-one mapping between the set of liquid equilibria and that of the sequences Λ that satisfy (24),(25), and $L \leq \phi_1(b_1 + x_1)(P_M + \alpha_M)$. The equilibria are as described in Proposition 3 except when $[\phi_1(b_1 + x_1) - b_0](P_M + \alpha_M) < L \leq \phi_1(b_1 + x_1)(P_M + \alpha_M)$. In this case the equilibrium is such that M accommodates as much as possible: $P_1 = P_M + \alpha_M$, and F does not spend all of $\phi_0 b_0$ at date 0, but uses part of it to extinguish L instead.*

Proof. See Appendix B.2. □

Default as an alternative to hyperinflation. The fiscal authority's cost from sovereign default α_F plays no role in equilibrium determination. This owes to the assumption that it is larger than the maximum gross resources that F could generate by defaulting at any date ($\alpha_F > 1$). This implies that default in equilibrium is always triggered by M, which does so when all the current real resources of the public sector do not suffice to repay the outstanding liabilities without a rate of inflation larger than α_M/P_M. In other words, default in this economy is a strategic decision of the central bank, who prefers this option to the always available one of letting the price level rise without limits — that is, hyperinflation.

The smaller α_M, the less M chickens out. Unsurprisingly, the strategy of fiscal irresponsibility of F is less effective, the less M cares about default. The new equilibrium feature introduced by finite default costs is that M is not willing to generate whichever date-0 inflation averts sovereign default. Its indifference point between inflation and outright default is at the price level $P_1 = P_M + \alpha_M$. Anticipating this, F cannot spend its entire initial surplus $\phi_0 b_0$ when L is sufficiently large. It must instead reimburse some of L at date 0 so as to ensure that M is exactly at its indifference point at date 1. In particular, Proposition 5 confirms that strategic fiscal irresponsibility is totally ineffective if $\alpha_M = 0$.

3.5.2 Illiquid equilibria

Our restriction to liquid equilibria simplifies the exposition but is admittedly arbitrary. Here we sketch how lifting it affects the analysis. Appendix B.2 develops the full-fledged analysis. Unlike under the restriction to liquid equilibria, there may now be several equilibria associated with a given sequence $\Lambda = (b_t, x_t)_{t\geq -1}$ as soon as the real demand for reserves x_t is equal to 0 for some dates $t \geq 0$.[12] Let $\Lambda = (b_t, x_t)_{t\geq -1}$ a sequence of positive numbers such that $b_{-1} = x_{-1} = 0$ and for all $t \geq 0$,

$$b_{t-1} + x_{t-1} \leq \phi_t(b_t + x_t) \leq 1. \tag{27}$$

Also recall that λ_t denotes the net aggregate real flow (20) received from savers at date t. The equilibria associated with Λ have the following features:

- If $x_0 = b_0 = x_1 = b_1 = 0$ then the public sector defaults at date 1 (but not afterwards). Otherwise equilibria are default-free and:
- If $\lambda_t = 0$ for some $t > 1$, then for some histories of the game up to t, any price $P_t \geq P_M$ is an equilibrium outcome. There are also histories, detailed in Appendix B.2, such that M can impose $P_t = P_M$.
- If $x_0 = \lambda_1 = 0$, then any price above $\max\{P_M; L/b_0\}$ is an equilibrium outcome;
- Otherwise all equilibria associated with Λ are identical at date t, at which date they have the same features as liquid equilibria.

Two new features arise when the demand for reserves may dry up at any date. First, if the public sector is completely illiquid before date 1, then it has no choice but defaulting at date 1. Second and more interestingly, the possibility that M does not receive fresh liquidity at every given date creates room for equilibria with higher price levels than P_M. The important bottom line though is that even when Λ is associated with multiple equilibria, the comparative statics with respect to b_0 in Proposition 4 still apply.

3.5.3 Sticky prices

In our analysis, the game of chicken is between a central bank that targets a price level and the government. In sticky-price environments such as the new-Keynesian model, the central bank imperfectly controls the sequence of real rates $(\{r_t\}_{t\geq 0})$ by setting nominal rates with the objective of targeting a sequence of natural interest rates $(\{r_t^n\}_{t\geq 0})$. This sequence of real rates affects the government's budget constraint and, in turn, the

[12]This is the counterpart under pleasant arithmetic of the situation in which $(b, x) = (\tau, 0)$ under an unpleasant one.

government's risk of default may force the central bank to set nominal rates consistent with real rates below the central bank's objective ($r_t < r_t^n$) to relax government's funding conditions. This also defines a game of chicken that is homeomorphic to the one that we consider.

4 General conditions for a pleasant monetary arithmetic

The goal of this section is to offer a general formalization of the idea that public liquidity supply can make the monetary arithmetic pleasant. The aim is to show that our insights are not specific to the OLG setup used in Sections 2 and 3. They carry over in any environment in which unbacked public liabilities carry a sufficiently high convenience yield over other consumption claims. To this purpose, we first introduce a general setting. In this setting, we define and characterize a pleasant monetary arithmetic, and show how such pleasantness affects the set of feasible price levels that the monetary authority can target. We then extend our results to multiple public liabilities – government debt and central bank reserves.

4.1 General setup

A policy (P, s) is a pair of sequences of real numbers $P = (P_t)_{t\in\mathbb{N}}$ and $s = (s_t)_{t\in\mathbb{N}}$. The sequence P represents the public sector's choice of price levels, and s stands for the sequence of real fiscal surpluses. We seek to identify the set of feasible policies of an economy, that is, the policies (P, s) that are compatible with market clearing and optimization by the private sector. We formally proceed as follows. Let $D_{-1} > 0$. For every $t \in \mathbb{N}$, let ϕ_t be a mapping from $\mathbb{R}^+$ into $\mathbb{R}^+ \setminus \{0\}$, $\bar{\phi}_t \le \phi_t$ a mapping from $\mathbb{R}^+$ into itself, and S_t a convex subset of $\mathbb{R}$ containing 0.

Definition 2. ***(Feasible policy)*** *Given* $\left(D_{-1}, \left(S_t, \phi_t, \bar{\phi}_t\right)_{t\in\mathbb{N}}\right)$, *a policy* (P, s) *is feasible if*

(i) For all $t \in \mathbb{N}$, $P_t > 0$ *and* $s_t \in S_t$.

(ii) There exists a sequence of real numbers $(D_t)_{t\in\mathbb{N}}$ *that satisfies for all* $t \in \mathbb{N}$:

$$\frac{D_{t-1}}{P_t} = s_t + \phi_t\left(\frac{D_t}{P_{t+1}}\right)\frac{D_t}{P_{t+1}}, \tag{28}$$

$$\lim_{\tau\to\infty}\left[\left(\prod_{i=t}^{\tau}\bar{\phi}_i\left(\frac{D_i}{P_{i+1}}\right)\right)\frac{D_\tau}{P_{\tau+1}}\right] = 0. \tag{29}$$

We denote by $\mathcal{F}$ the set of such feasible policies.

The above abstract definition subsumes feasible fiscal and monetary policies in many models. The parameter D_{-1} represents nominal public liabilities inherited from an unmodelled past and due at date 0. It is the counterpart of L in Sections 2 and 3 up to the difference that here D_{-1} is due at date 0 whereas L was due at date 1.[13] For every $t \geq 0$, D_t corresponds to a nominal liability—a number of currency units due at date $t+1$ and issued at date t by the public sector. Conditions (28) and (29) are reduced forms for the restrictions that optimization by the private sector and market clearing impose on feasible fiscal policies s and monetary policies P in a large class of standard economic models. Condition (28) is the date-t budget constraint of the public sector given the pricing ϕ_t of public debt, and (29) is a transversality condition.

In other words, our approach consists in summarizing an economy with the two ingredients that matter for the determination of feasible fiscal and monetary policies, the pricing of public debt $(\phi_t)_{t\in\mathbb{N}}$ and the intertemporal rates of substitution in the transversality condition $(\bar{\phi}_t)_{t\in\mathbb{N}}$. If $\bar{\phi}_t = \phi_t$ then public debt does not offer liquidity services. The OLG setup used in Sections 2 and 3 is a first example of an economy to which this general setting applies. We also introduce in this section a second simple example in which an infinitely-lived representative agent directly derives utility from its current holdings of public liabilities:

OLG example. Assuming $\bar{\tau}_1 = 0$, we get $\phi_t(d) = \min\{e^{-\rho}; 1/d\}$ when t does not belong to $\{1 + k(T+1), k \in \mathbb{N}\}$ and $\phi_t(d) = \min\{e^{\delta}; 1/d\}$ otherwise. The function $\bar{\phi}_t$ can be normalized to 0 so that (29) always holds given that optimization by short-lived agents requires no transversality condition. The set S_t is $(-\infty, 0]$.

Debt in the utility function. Consider an economy populated by a public sector and a private sector comprised of a representative agent. The agent receives $Y > 0$ consumption units at each date $t \in \mathbb{N}$. She derives utility out of consumption and real holdings of public liabilities. At each date $t \in \mathbb{N}$, the public sector sets the price level P_t, raises a real lump-sum tax s_t, and issues a nominal claim of D_t currency units due at $t+1$. It starts out with an exogenous legacy nominal liability $D_{-1} > 0$ due at date 0.

Denoting q_t the date-t real price of public bonds, the agent selects a consumption

[13] The analysis is verbatim no matter when D_{-1} is due, up to the difference that only the price level at the due date matters. So here only P_0 is constrained by fiscal policy whereas so was P_1 in Sections 2 and 3.

stream $(C_t)_{t\in\mathbb{N}}$ and bond holdings $(H_t)_{t\in\mathbb{N}}$ that solve for some $\beta \in (0,1)$

$$\max_{(C_t,H_t)_{t\in\mathbb{N}}} \sum_{t=0}^{\infty} \beta^t \left(u(C_t) + v\left(\frac{H_t}{P_{t+1}}\right)\right),$$
$$\text{s.t. } \forall t \in \mathbb{N}, Y_t - s_t + \frac{H_{t-1}}{P_t} = q_t \frac{H_t}{P_{t+1}} + C_t. \tag{30}$$

In addition, goods and bonds market clearing implies that for all t,

$$C_t = Y \text{ and } H_t = D_t. \tag{31}$$

Standard restrictions on u and v (e.g., Kamihigashi, 2003) imply that $(C_t, H_t)_{t\in\mathbb{N}}$ solves (30) and (31) if and only if it satisfies two conditions, a local one (Euler equation):

$$q_t = \frac{\beta u'(C_{t+1}) + v'\left(\frac{H_t}{P_{t+1}}\right)}{u'(C_t)}, \tag{32}$$

and a terminal one (transversality condition):

$$\lim_{k\to\infty} \left[\prod_{t=0}^{k} \frac{\beta u'(C_{t+1})}{u'(C_t)}\right] \frac{H_k}{P_{k+1}} = 0. \tag{33}$$

Injecting (31) yields $\phi_t(d) = \beta + \frac{v'(d)}{u'(Y)}$ and $\bar{\phi}_t = \beta$. The set S_t is $\mathbb{R}$ for any $t \geq 0$.

We impose in the remainder of the paper the following restrictions.

Assumption 1. *For every $t \in \mathbb{N}$, the function ϕ_t is continuously decreasing, the function $d \mapsto \bar{\phi}_t(d)/\phi_t(d)$ is increasing and the function $d \mapsto d\phi_t(d)$ is strictly increasing.*

The first item of Assumption 1 imposes that, at any date, the price $\phi_t(d)$ decreases with respect to the real level of the newly issued debt d. The second item imposes that this decrease dominates a potential decrease in $\bar{\phi}(.)$.[14] Finally, the third item imposes that even though the price of debt decreases with the quantity of newly issued debt, the real amount of fiscal resources increases with it. We will see that the debt-in-the-utility example satisfies these restrictions for standard specifications for v. The OLG example also satisfies the first two restrictions, but since the saving capacity is bounded above by the endowment, the mapping $d \mapsto d\phi_t(d)$ is only strictly increasing when d takes values in a bounded interval. We will highlight in the following when this affects our results.

[14] As we will see shortly, this basically means that the convenience yield on public debt decreases with the level of newly issued debt.

4.2 Pleasant monetary arithmetic: Definition and characterization

We now investigate the interdependence between fiscal and monetary policy in the general setting that we have introduced. To this purpose, we first formalize the notion of a tradeoff between these two policies. We then connect the presence of tradeoffs to the price of public debt and to whether a rollover of debt is possible.

Fiscal-monetary tradeoff A feasible policy features a fiscal-monetary tradeoff—simply a "tradeoff" henceforth— if and only if there exists no other feasible policy with both smaller surpluses and initial price level. Formally,

Definition 3. ***(Tradeoff)*** *A feasible policy (P, s) features a tradeoff if and only if for any policy $(P', s') \in \mathcal{F}$,*

$$P'_0 \leq P_0 \text{ and } s' \leq s \rightarrow P'_0 = P_0 \text{ and } s' = s. \tag{34}$$

Starting from a policy that features a tradeoff, a monetary authority averse to sovereign default would have to accommodate with inflation if the fiscal one were willing to reduce taxes, and vice versa. The following proposition establishes an equivalence between the presence of a fiscal-monetary tradeoff and the extent to which fiscal policy uniquely pins down the price level.

Proposition 6. ***(A tradeoff implies fiscal determination of the price level)***
Let $(P, s) \in \mathcal{F}$ and $\mathcal{P}_s = \{P'_0 \mid (P', s) \in \mathcal{F}\}$.

(i) *$\mathcal{P}_s$ is an interval;*

(ii) *$\mathcal{P}_s$ is a singleton if and only if every policy $(P', s) \in \mathcal{F}$ features a tradeoff.*

Proof. See Appendix B.3. □

Proposition 6 first shows that the set of feasible initial price levels associated with a given path of surpluses is an interval. This owes to the monotonicity posited by Assumption 1. It then establishes the equivalence between the existence of a tradeoff for every feasible policy (P', s) and the fact that s fully determines the initial price level.

Pleasant monetary arithmetic. Based on our definition of a tradeoff, we now offer a more global concept of fiscal and monetary interdependence.

Definition 4. ***(Pleasant monetary arithmetic)*** *Given $(S_t, \phi_t, \bar{\phi}_t)_{t\in\mathbb{N}}$, the monetary arithmetic is unpleasant if for all $D_{-1} > 0$, every feasible policy features a tradeoff. Otherwise, the monetary arithmetic is pleasant.*

The monetary arithmetic is unpleasant when any feasible path for public finances features a tradeoff. The arithmetic is conversely pleasant as soon as there exists an initial value of legacy liabilities $D_{-1} > 0$ for which fiscal and monetary policy are not interdependent this way.

We now characterize such situations of pleasant arithmetic. Our characterization requires the introduction of debt rollovers, which requires in turn that of the convenience yield on public liabilities.

Definition 5. ***(Convenience yield)*** *We deem $\delta_t(d)$ the date-t convenience yield associated with real debt $d > 0$ defined as*

$$\delta_t(d) = \begin{cases} \log(\phi_t(d)) - \log(\bar{\phi}_t(d)) & \textit{if } \bar{\phi}_t(d) > 0, \\ 1 & \textit{otherwise.} \end{cases} \tag{35}$$

We are now equipped to define debt rollovers.

Definition 6. ***(Rollover)*** *A rollover is a sequence $(d_t)_{t\in\mathbb{N}} \in (0, +\infty)^{\mathbb{N}}$ such that for all $t \in \mathbb{N}$,*

$$d_t = d_{t+1}\phi_t(d_{t+1}), \tag{36}$$

and such that $\sum_{t\in\mathbb{N}} \delta_t(d_t)$ diverges.

The existence of such a rollover is a sufficient and necessary condition for the monetary arithmetic to be pleasant:

Proposition 7. ***(Characterization of pleasant monetary arithmetic)*** *The monetary arithmetic is pleasant if and only if there exists a rollover.*

Proof. See Appendix B.4. □

Put simply, if public liabilities can be rolled over at a cost that is sufficiently low relative to the long-term opportunity cost of capital of the private sector, then the public sector can extract the resulting surplus in order to gain degrees of freedom and relax the interdependence between fiscal and monetary policies.

We actually show in the proof of Proposition 7 that if there exists such a rollover, then the set of initial prices associated with zero primary surpluses ($s = 0$), $\mathcal{P}_0$, is not a singleton regardless of the initial level of debt D_{-1}.

Proposition 8. ***(Fiscal determination at the minimum price level)*** *For every* $D_{-1} > 0$,

$$s' < s \to \inf \mathcal{P}_{s'} > \inf \mathcal{P}_s. \tag{37}$$

Proof. See Appendix B.5. □

Many observers hold the view that fiscal considerations matter for monetary policy only in times of stretched public finances, or, that the fiscal theory of the price level is practically relevant only during fiscal crises. Proposition 8 translates this in our formal setting. The proposition states that even when surpluses do not dictate the price level, there is still a point at which the tradeoff between a lower price level and lower surpluses kicks in again. In this sense the independence of the central bank is not warranted in the absence of limitations to fiscal irresponsibility.

A necessary condition for Proposition 8 is that $d \mapsto d\phi_t(d)$ is strictly increasing for any positive d: the public sector is always able to raise resources. Such a condition is not satisfied by our OLG example where $d \mapsto d\phi_t(d)$ is strictly increasing only on an interval before being constant. In this case, the inequality in the right-hand side of (37) is not strict. Indeed, it may be the case that the minimum price level is unaffected by a monotonous change in the stream of surpluses: future additional resources cannot be transferred to today through borrowing.

It is instructive to connect the results in Propositions 7 and 8 to the OLG and the debt-in-the-utility examples.

OLG example. In this case, $\bar{\phi}_t = 0$ for all t and so the monetary arithmetic is pleasant if and only if there exists a rollover. As established in Section 2, this is the case if and only if $T \leq \delta/\rho$. Regarding Proposition 8, the function $d \mapsto d\phi_t(d)$ is constant above some threshold and so the minimum date-1 price level is equal to L no matter the value of s_1.

DIU example. In this case, $\bar{\phi}_t$ is a constant equal to β, and ϕ_t is given by the right-hand side of (32). For all $t \in \mathbb{N}$, $S_t = \mathbb{R}$. We study here the case in which

$$v(h) = \alpha u'(Y) \frac{h^{1-\gamma}}{1-\gamma}, \tag{38}$$

where $\alpha > 0$ and $\gamma \in (0, 1)$. The existence of a rollover is not an issue in this case since

$$d\phi_t(d) \equiv \Psi(d) = \beta d + \alpha d^{1-\gamma} \tag{39}$$

is a bijection over $\mathbb{R}_+$. In fact, it is possible to construct a rollover starting from any positive initial value. The question is then whether there exists one with a diverging sum of convenience yields. We have

$$\forall t \in \mathbb{N}, \delta_t(d) = \log\left(1 + \frac{\alpha d^{-\gamma}}{\beta}\right). \tag{40}$$

Let

$$d^* = \left(\frac{\alpha}{1-\beta}\right)^{\frac{1}{\gamma}} \tag{41}$$

the unique non-zero fixed point of Ψ. One can show that a rollover has a diverging sum of convenience yields if and only if $d_0 \in (0, d^*]$. For such values of d_0 the rollover converges to a finite value, whereas for $d_0 > d^*$ it tends to $+\infty$ so quickly that the convenience yields decrease sufficiently fast for $\sum \delta_t$ to converge .

This implies first that there exists a rollover with a diverging sum of convenience yields, and so that the monetary arithmetic is pleasant. This also implies that for a given surplus stream of the form $s = \left(\mathbb{1}_{\{t=0\}} x\right)_{t \in \mathbb{N}}$ for some $x > 0$, we have

$$\inf \mathcal{P}_s = \frac{D_{-1}}{x + d^*}, \tag{42}$$

which decreases in x and in the preference for liquidity α, and increases in the elasticity of the convenience yield γ. This example thus generates the interesting insight that the interdependence between fiscal and monetary policy is tighter when the demand for public liquidity is more elastic. In particular, in the limiting case $\gamma = 1$, that is $v(h) = \alpha \log h$, where $\alpha > 0$, then, (30), (32) and (33) lead to

$$\frac{D_{-1}}{P_0} = \sum_{t \in \mathbb{N}} \beta^t s_t + \frac{\alpha}{u'(Y)(1-\beta)}. \tag{43}$$

This means that there is a strong interdependence between monetary and fiscal policies: As in the fiscal theory of the price level, the present value of future surpluses uniquely pins down the price level, but at a lower level compared with the fiscal theory because of the convenience yield (second term on the right-hand side of (43)). One can observe that a large value to hold public liabilities (as measured by the parameter α) leads to a lower price level P_0 through equation (43).

4.3 Extension to multiple public liabilities

This subsection extends the analysis of feasible policies to the case where the public sector may issue two types of liabilities, government debt and central-bank reserves, and where the central bank and the government have separate budget constraint as in Bassetto and Messer (2013) or Hall and Reis (2015).

Feasible policies with central-bank reserves. We now assume that the public liabilities consist in both government bonds and central-bank remunerated reserves. We denote by $X_t \geq 0$ ($B_t \geq 0$) the amount of reserves (bonds) issued at date t and due at date $t+1$ and by ϕ_t^X (ϕ_t) the real price of reserves (bonds).[15] We suppose that ϕ_t^X and the price of bonds ϕ_t may a priori depend on the real holdings of both bonds and reserves by the private sector.

The monetary authority may use the proceeds from issuing reserves to trade government bonds or/and transfer all or part of them to the government. Let B_t^B denote the date-t debt holding of the central bank, and θ_t denote the date-t real remittances to the government. These transfers can be positive (dividends) or negative (recapitalization).

Definition 7. ***(Feasible extended policy)*** *Given* $\left(B_{-1}, B_{-1}^B, X_{-1}, (S_t, \phi_t, \phi_t^X, \bar{\phi}_t)_{t\in\mathbb{N}}\right)$, *an extended policy* (P, s, θ) *is feasible if*

(i) For all $t \in \mathbb{N}$, $P_t > 0$ *and* $s_t \in S_t$.

(ii) There exists a triplet of sequences of positive real numbers $(B_t, X_t, B_t^B)_{t\in\mathbb{N}}$ *that satisfies for all* $t \in \mathbb{N}$:

$$\frac{B_{t-1}}{P_t} = s_t + \theta_t + \phi_t((B_t - B_t^B)/P_{t+1}, X_t/P_{t+1})\frac{B_t}{P_{t+1}}, \tag{44}$$

$$\phi_t((B_t - B_t^B)/P_{t+1}, X_t/P_{t+1})\frac{B_t^B}{P_{t+1}} + \frac{X_{t-1}}{P_t} + \theta_t = \frac{B_{t-1}^B}{P_t} + \phi_t^X((B_t - B_t^B)/P_{t+1}, X_t/P_{t+1})\frac{X_t}{P_{t+1}}, \tag{45}$$

$$\lim_{\tau\to\infty}\left(\prod_{i=t}^{\tau}\bar{\phi}_i((B_i - B_i^B + X_i)/P_{i+1})\right)\frac{B_\tau - B_\tau^B + X_\tau}{P_{\tau+1}} = 0, \tag{46}$$

$$B_t^B \leq B_t. \tag{47}$$

We denote by $\mathcal{F}^X$ *the set of such feasible extended policies.*

[15] For expositional simplicity and symmetry, we model central-bank reserves as one-period claims akin to government bonds. Introducing reserves with indefinite maturity as they are in practice would not add any insight.

Equation (44) is simply the budget constraint of the government (28) where remittances from the central bank are added to surpluses. Equation (45) is the budget constraint of the central bank. Equation (46) is the terminal condition that only depends on the private holding of public liabilities $B_t - B_t^B + X_t$. Finally, equation (47) imposes that the bonds' holding of the central bank be lower than the total stock of government bonds.

For a given stream of surpluses s, we denote by $\mathcal{P}_s^X$ the set of initial price levels that belong to a feasible extended policy: $\mathcal{P}_s^X = \{P_0 | (P, s, \theta) \in \mathcal{F}^X\}$. We now study how the availability of such extended policies expands the set of circumstances in which monetary arithmetic is pleasant. We study in turn the polar cases in which only reserves carry a convenience yield, and in which bonds and reserves are perfect substitutes.

When bonds and reserves are imperfect substitutes. Suppose that reserves are the only asset providing liquidity services leading to a disconnect between the price of reserves and that of bonds. For simplicity, we assume that $\phi_t = \bar{\phi}_t$ does not depend on the quantity of public liabilities and that ϕ_t^X only depends on real holdings of reserves. More precisely, at any date t:

$$\phi_t = \bar{\phi}_t \text{ and } \phi_t^X(X_t/P_{t+1}) \geq \phi_t \text{ with } \phi_t^X(0) > \phi_t. \tag{48}$$

In this context, it is useful to extend our definitions of rollover and convenience yield to central bank's reserves. A *rollover for reserves* is a sequence $(x_t)_{t\in\mathbb{N}} \in (0, +\infty)^{\mathbb{N}}$ such that for all $t \in \mathbb{N}$,

$$x_t = x_{t+1}\phi_t^X(x_{t+1}). \tag{49}$$

We deem $\delta_t^X(x)$ the date-t convenience yield associated with real reserves $x > 0$ defined as

$$\delta_t^X(x) = \begin{cases} \log(\phi_t^X(x)) - \log(\bar{\phi}_t) & \text{if } \bar{\phi}_t > 0, \\ 1 & \textit{otherwise.} \end{cases} \tag{50}$$

Importantly, a rollover for reserves is expressed in real terms. In nominal terms, reserves can always be rolled over, as reserves are always reimbursed using reserves.

Proposition 9. ***(Central bank's issuance of reserves may relax the fiscal-monetary tradeoff)*** *Suppose $X_{-1} = B_{-1}^B$. If there exists a rollover of reserves $\{x_t\}_{t\geq 0}$ such that $\sum_{t\in\mathbb{N}} \delta_t^X(x_t)$ diverges, then for any feasible policy (P, s), there exists a feasible extended policy (P', s', θ) with $P_0 < P_0'$ and $s \leq s'$.*

Proof. See Appendix B.6. □

When fiscally unbacked reserves can be rolled-over then the issuance of reserves generates additional real revenue for the central bank. Whether it is through monetary financing of the deficit or through open market operations, these additional revenue are ultimately rebated to the government leading to either a lower price level or lower surpluses. The existence of a convenience yield on reserves thus potentially relaxes the interdependence between the two authorities.

Remark. Proposition 9 contrasts with the unpleasant monetary arithmetic described by Sargent and Wallace (1981). In their paper, a larger deficit necessarily leads to a higher price level. Indeed, when the government increases deficits, the only feasible monetary policy is to raise the stock of money eventually collecting higher seigneurage revenues to balance the government budget. A quantity theory of money then links the quantity of money and the price level and therefore larger deficits cause higher prices. Because, we do not assume such a link here, the central bank can increase reserves to generate additional revenues without triggering a higher price level.

When bonds and reserves are perfect substitutes. Let us now turn to the case where bonds and reserves are perfect substitutes. This happens when they share the same price that depends on the total public liability held by the private sector, $D_t = B_t - B_t^B + X_t$, that is:

$$\text{For all } t \in \mathbb{N},\ \phi_t^X(D_t) = \phi_t(D_t). \tag{51}$$

The budget constraint of the central bank and that of the government can then be consolidated without loss of generality:

$$\frac{D_{t-1}}{P_t} = s_t + \phi_t(D_t/P_{t+1})\frac{D_t}{P_{t+1}}. \tag{52}$$

It follows that only the path of aggregate public liabilities held by the private sector D_t matters for price-level determination.

Proposition 10. ***(Central bank's balance sheet irrelevance for price determination)*** *If reserves and bonds are perfect substitutes, then for a given level of public liability held by the private sector D_{-1} and fiscal surpluses s, the set of initial feasible prices is unaffected by the central bank's balance-sheet structure: $\mathcal{P}_s = \mathcal{P}_s^X$.*

Proof. See Appendix B.7. □

Proposition 10 shows that, given surpluses, the set of initial feasible prices is unaffected by the balance-sheet tools of the central bank. In particular, the path of remittances θ is irrelevant for the determination of the price level because the government can adjust the issuance of debt such as to exactly offset the newly issued reserves, letting the quantity of debt held by the public unchanged. Therefore, when reserves and bonds are perfect substitutes, monetary financing of the deficit has no impact on the price level. Note that this is true regardless of whether the monetary arithmetic is pleasant or not.

In addition, open-market operations at any date (defined here as the case $X_t = D_t^B$) do not change the public liability D and do not affect the set of feasible policies (P, s). This result echoes the well-known irrelevance result of open market operations (Wallace, 1981; Chamley and Polemarchakis, 1984). Indeed, if reserves and government bonds do not provide different liquidity services then open market operations do not alter the aggregate budget constraint of the public sector vis-à-vis the private sector.

An implication of Proposition 10 is that when a rollover is impossible (see Proposition 7), then the only initial price level that is feasible is the singleton $\mathcal{P}_s$ that only depends on the legacy debt D_{-1} and the stream of surpluses s but is completely independent of the amount of reserves and of central bank's debts holding. In a similar environment, Benigno (2017) establishes the related result that if the government passively passes through the remittances to the public (in our context, if $s = -\theta$), the central bank can pin down the price level because it de facto decides on the stream of fiscal surpluses.

5 Concluding remarks

This paper contributes to the study of fiscal and monetary interactions in two ways. We first offer fairly general conditions under which the public sector's ability to meet private demand for liquidity relaxes fiscal and monetary interdependence. The proceeds that the public sector may generate from supplying liquidity expands the set of jointly feasible fiscal and monetary policies, so much so that several price levels can be compatible with a given path of fiscal surpluses.

We then contend that this latter result need not imply that liquidity supply reinforces central-bank independence. Assessing this independence requires a model that takes into account the very reason an independent central bank is needed in the first place: lack of commitment by the fiscal authority. We write down such a model, and this leads us to conclude that it is the monetary authority's ability to preempt liquidity demand, rather than the magnitude of this demand per se, that warrants the independence of the central bank.

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Appendix

A Extension: multiple legacy liabilities

We extend subsection 2.3 by assuming that there are two legacy liabilities B_{-1} and L maturing at dates 0 and 1 respectively. This affects the date-0 budget constraint of the fiscal authority (8):

$$B_{-1} = \theta_0 + \sigma_0 + \tau_0 + \phi_0 \left(\frac{B_0}{P_1} - \frac{L_F}{P_1} \right), \tag{53}$$

leading to consolidated budget constraints at dates 0 and 1:

$$\frac{B_{-1}}{P_0} = \sigma_0 + \tau_0 + \phi_0 \frac{B_0 + X_0 - L_M - L_F}{P_1}, \tag{54}$$

$$\frac{X_0 + B_0 + L - L_F - L_M}{P_1} = \sigma_1 + \tau_1 + \omega, \tag{55}$$

where $\omega \in [0, 1 - \tau_1]$ if the monetary arithmetic is pleasant and 0 otherwise.

Unpleasant monetary arithmetic Suppose $T > \delta/\rho$. Then (P_0, P_1) are constrained by the intertemporal budget constraint:

$$\frac{B_{-1}}{P_0} + \phi_0 \frac{L}{P_1} = \sigma_0 + \tau_0 + \phi_0(\sigma_1 + \tau_1). \tag{56}$$

As in Cochrane (2001), the exact split between P_0 and P_1 however depends on the issuance policy at date 0:

$$P_0 = \frac{B_{-1}}{\sigma_0 + \tau_0 + \phi_0(\sigma_1 + \tau_1)\frac{D_0 - L_{PS}}{L + D_0 - L_{PS}}}, \tag{57}$$

$$P_1 = \frac{L + (D_0 - L_{PS})}{\sigma_1 + \tau_1}. \tag{58}$$

In our context of multiple liabilities and for a given stream of fiscal surpluses, the key determinant of the price pattern is the date-0 debt issuance that is not used to buyback L, that is, $D_0 - L_{PS} = B_0 - L_F + X_0 - L_M$, that is jointly determined by fiscal and monetary authorities. The higher the net issuance of new debt $D_0 - L_{PS}$, the lower the date-0 price level is relative to the date-1 price level. Therefore, the higher $B_0 - L_F$, the smaller the feasible set of prices and especially the higher the date-1 price level is.

Pleasant monetary arithmetic Suppose now $T \leq \delta/\rho$. Then (P_0, P_1) are not fully constrained by the intertemporal budget constraint:

$$\frac{B_{-1}}{P_0} + \phi_0 \frac{L}{P_1} = \sigma_0 + \tau_0 + \phi_0(\sigma_1 + \tau_1) + \omega, \text{ with } \omega \in [0, 1 - \tau_1]. \tag{59}$$

In addition, as in the unpleasant case, the split between P_0 and P_1 is defined by the issuance of new public debt at date 0 that is not used to buyback share or all part of L.

$$P_0 = \frac{B_{-1}}{\sigma_0 + \tau_0 + \phi_0(\sigma_1 + \tau_1 + \omega)\frac{D_0 - L_{PS}}{L + D_0 - L_{PS}}}, \tag{60}$$

$$P_1 = \frac{L + (D_0 - L_{PS})}{\sigma_1 + \tau_1 + \omega}. \tag{61}$$

B Proofs

B.1 Proof of Proposition 1

After date 2. F and M can only issue securities at date 0 that are backed by the date-1 fiscal capacity $\bar{\tau}_1$. Thus there is no action after date 1 other than M setting the price level at P_M.

No default in equilibrium. Suppose an equilibrium is such that the public sector defaults at date 1. Default should be total given the fixed cost of doing so, and thus liquidity demand such that $\bar{X}_0 = \bar{B}_0 = 0$. But in this case M would set $P_1 = L/\bar{\tau}_1$ and avoid default, a contradiction.

Equilibria when $(b, x) \neq (\bar{\tau}_1, 0)$. We show that there is exactly one equilibrium associated with such (b, x). We denote $\bar{P}_1$ the savers' anticipation of the date-1 price at date 0, X and B the respective (nominal) supply of liquidity by M and F at date 0, and L_M and L_F the respective (nominal) shares of L that they prepay at date 0. An equilibrium is such that

$$\bar{X}_0 = x\bar{P}_1, \tag{62}$$

$$\bar{B}_0 = b\bar{P}_1, \tag{63}$$

$$0 \leq L_M \leq X \leq \bar{X}_0, \tag{64}$$

$$0 \leq L_F \leq B \leq \bar{B}_0, \tag{65}$$

$$0 \leq L_F + L_M \leq L. \tag{66}$$

Moving first at date 1, M sets P_1 at the largest of two values, either P_M or the smallest P_1 that ensures solvency:

$$P_1\bar{\tau}_1 \geq L - L_F - L_M + X + B. \tag{67}$$

Rationally anticipating this at date 0, F sets $B = \bar{B}_0$ and $L_F = 0$ so as to maximize $B - L_F$, date-0 transfers to young entrepreneurs, σ_0, and the date-1 price level P_1. On the contrary, M sets $X - L_M = 0$ to reduce as much as possible the date-1 price level P_1. Thus

$$P_1 = \max\left\{P_M; \frac{L + \bar{B}_0}{\bar{\tau}_1}\right\}, \tag{68}$$

leading to a unique rational expectations price $\bar{P}_1 = P_1$:

$$P_1 = \max\left\{P_M; \frac{L}{\bar{\tau}_1 - b}\right\}. \tag{69}$$

Equilibria when $(b, x) = (\bar{\tau}_1, 0)$**.** In this case,

$$\bar{\tau}_1 P_1 \geq L - L_F + \bar{\tau}_1\bar{P}_1 \tag{70}$$

implies that equilibria must be such that $L_F = L$. In this case, any savers' beliefs $P_1 \geq \max\{P_M; L/\bar{\tau}_1\}$ can be sustained in equilibrium. Indeed, in addition to be at least larger than P_M, the only condition that P_1 has to satisfy is $b\bar{P}_1 \geq L_F = L$ with $b = \bar{\tau}_1$ and so $P_1 \geq L/\bar{\tau}_1$.

B.2 Proof of Propositions 3, 5, and of the results in Section 3.5.2

We characterize the equilibria in the general case in which $\alpha_M > 0$, $\alpha_F > 1$, and allowing for zero demand for reserves at any date—that is, without restricting the analysis to liquid equilibria. The results in Section 3.5.2 then obtain by letting $\alpha_F, \alpha_M \to +\infty$, that in Proposition 5 by restricting the analysis to equilibria such that $x_t > 0$ for all $t \geq 0$, and that in Proposition 3 by doing both.

Note first that any equilibrium must be such that $P_t \geq P_M$ for all $t \geq 2$. Suppose by contradiction that $P_t < P_M$. Then M can raise the price to P_M thereby both reaching its target and reducing the burden of debt repayment, and savers should anticipate this at $t - 1$.

At dates $t \geq 2$. First notice that $\lambda_t \geq 0$ because F cannot raise taxes for $t \geq 2$.

- If $\lambda_t > 0$, then the equilibrium must be such that $P_t = P_M$. Suppose by contradiction that $P_t > P_M$. Then M can reduce the price, possibly but not necessarily all the way to P_M, using all or part of λ_t to pay for the induced increase in real debt due. This is feasible because F always prefers diminishing transfers to entrepreneurs rather than default.

- If $\lambda_t = 0$, suppose that there exists $k \geq 1$ such that $x_{t-k}, x_{t-k+1}, ..., x_{t-1} > 0$ and $\lambda_{t-k} > 0$. (Note that such a k always exists when the equilibrium is liquid since $x_0 = \lambda_0 > 0$). In this case it must also be that $P_t = P_M$: Otherwise M can optimally leave an arbitrarily small fraction of liquidity demand $\bar{X}_{t-k}$ unsatisfied, and so on at each date until $t-1$ thereby ensuring $\lambda_t > 0$ in which case it can reduce the price as seen above. If such a k does not exist then any $P_t \in [P_M, P_M + \alpha_M]$ is a sustainable equilibrium outcome, unless of course $x_{t-1} = b_{t-1} = 0$ in which case M can enforce $P_t = P_M$ since there are no outstanding liabilities at date t.

At dates 0 and 1.

- Note first that the public sector defaults on L at date 1 if and only if its nominal liabilities satisfy $L > \phi_1(x_1 + b_1)(P_M + \alpha_M)$ – i.e. nominal liabilities exceed what the public sector's income from the bubble evaluated with the maximum price level tolerated by the central bank. In this case, $x_0 = b_0 = 0$. Default comes at fixed costs for both authorities and so default is total when it occurs, which savers anticipate at date 0.

Otherwise, in the absence of default at date 1,

- If $\phi_1(b_1 + x_1) = b_0$ then it must be that $x_0 = 0$ and $b_0 > 0$. Any date-1 price $P_1 \in [\max\{P_M; L/b_0\}, P_M + \alpha_M]$ is a sustainable equilibrium outcome in this case (see the reasoning in B.1 when $\bar{b}_0 = \bar{\tau}_1$), implying that it must be that $b_0 \geq L/(P_M + \alpha_M)$.

- Suppose $\phi_1(b_1 + x_1) > b_0$. It must be that $L - L_{PS} \leq \lambda_1(P_M + \alpha_M)$, and M sets the date-1 price at the minimum level that averts default, $P_1 = \max\{P_M; (L - L_{PS})/\lambda_1\}$. This implies in turn that at date 0, it is weakly dominant for M to maximize L_{PS} and thus to use x_0 to prepay as much of L as possible, i.e., $L_M = P_1 x_0$. Conversely, F seeks to induce the highest possible value of P_1 and thus only prepays the minimum amount that averts date-0 default, spending the residual on young date-(0) entrepreneurs. As a result,

$$P_1 = \max\{P_M; (L - L_F)/(\phi_1(x_1 + b_1) - b_0)\}, \tag{71}$$

and hence,

- If $L \leq [\phi_1(b_1 + x_1) - b_0]P_M$ then the date-1 price is P_M and F does not prepay any of L at date 0;
- If $[\phi_1(b_1 + x_1) - b_0]P_M \leq L \leq [\phi_1(b_1 + x_1) - b_0](P_M + \alpha_M)$ then the date-1 price is $L/[\phi_1(b_1 + x_1) - b_0]$ and F does not prepay any of L at date 0;
- If $[\phi_1(b_1 + x_1) - b_0](P_M + \alpha_M) \leq L \leq \phi_1(b_1 + x_1)(P_M + \alpha_M)$ then the date-1 price is $P_M + \alpha_M$ and F prepays $L_F = L - [\phi_1(b_1 + x_1) - b_0](P_M + \alpha_M)$ at date 0.

B.3 Proof of Proposition 6

$\mathcal{P}_s$ is a convex set. Let us show that the set of initial feasible prices given s, $\mathcal{P}_s$, is convex. Let $P_0 > P'_0$ two feasible price levels associated with price level paths $\{P_t\}_{t\geq 0}$ and $\{P'_t\}_{t\geq 0}$. Let us show that any price level $P''_0 = \alpha P'_0 + (1-\alpha)P_0$ with $\alpha \in [0, 1]$ is also feasible. To this purpose, let us build a sequence of price levels $\{P''_t\}_{t\geq 0}$ so that $P''_t = P'_t$ for any $t > 0$ and let us show that $\{P''_t, s_t\}_{t\geq 0}$ is a feasible policy.

Given this sequence of prices, we can use the budget constraint of the government to construct a sequence of nominal debt D''_{t+1} because $d \mapsto d\phi_t(d)$ is continuously increasing. We find that D''_t/P''_{t+1} is smaller than D'_t/P'_{t+1} but greater than D_t/P_{t+1} for any date $t \geq 0$.

Let us now show that the transversality condition is satisfied. For any $k > 0$, the government budget constraint leads to:

$$\bar{\phi}''_k \frac{D''_k/P''_{k+1}}{D''_{k-1}/P''_k} = \frac{\bar{\phi}''_k}{\phi''_k}\left(1 - \frac{s_k}{D''_{k-1}/P''_k}\right). \tag{72}$$

According to Assumption 1, $\frac{\bar{\phi}''_k}{\phi''_k} \leq \frac{\bar{\phi}'_k}{\phi'_k}$ because debt levels satisfy $D''_k \leq D'_k$. Therefore:

$$\bar{\phi}''_k \frac{D''_k/P''_{k+1}}{D''_{k-1}/P''_k} \leq \frac{\bar{\phi}'_k}{\phi'_k}\left(1 - \frac{s_k}{D'_{k-1}/P'_k}\right) = \bar{\phi}'_k \frac{D'_k/P'_{k+1}}{D'_{k-1}/P'_k}, \tag{73}$$

using the fact that $D''_{k-1}/P''_k \leq D'_{k-1}/P'_k$.

The transversality condition can be rewritten as the limit when T tends to $+\infty$ of

$$\frac{D_{-1}}{P''_0} \prod_{k=0}^{T} \bar{\phi}''_k \frac{D''_k/P''_{k+1}}{D''_{k-1}/P''_k}. \tag{74}$$

Therefore inequality (73) shows that if the transversality condition is verified for the policy (P', s) it is also verified for (P'', s).

Given that the set of convex sets of real numbers are intervals, the set of initial prices is an interval of real numbers.

$\mathcal{P}_s$ and tradeoffs. Suppose that $\mathcal{P}_s$ is not a singleton. Then it is immediate that we can find a policy $U' \in \mathcal{F}$ that violates (34) (with $s' = s$ and $P'_0 < P_0$).

To show the converse implication ii), suppose that the feasible policy U does not feature a tradeoff: There exists a feasible policy $U' \neq U$ in $\mathcal{F}$ that violates (34). Suppose first that U' is such that $P'_0 < P_0$ and $s' \leq s$. We want to show that the policy $U'' = (P', s)$ is feasible: $U'' \in \mathcal{F}$. Define $\{D'_t\}_{t\geq 0}$ and $\{D''_t\}_{t\geq 0}$ the debt paths associated with U' and U''. Because $d \mapsto d\phi_t(d)$ is continuously increasing, D''_t exists and is smaller than D'_t. Besides the transversality condition is satisfied for the same reason as the one developed above (see equation (73)), and therefore $U'' \in \mathcal{F}$.

Suppose then that U' is such that $P'_0 = P_0$ and that there exists $\tau \geq 0$ such that $s'_\tau < s_\tau$ (and $s'_t \leq s_t$ otherwise). First, the same reasoning as above leads to prove that there exists a feasible policy U'' featuring $P'_0 = P_0$, $s'_\tau < s_\tau$, and $s'_t = s_t$ otherwise. Then, consider the problem at date $t = \tau$. One can raise the surplus at date τ to remain below s_τ and decrease the surplus at date $\tau - 1$ by issuing more debt—which is possible given that $d \mapsto d\phi_t(d)$ is strictly increasing. By doing so down to $t = 0$ one sees that there exists a trajectory of surpluses such that $s''_0 < s_0$ consistent with a feasible policy. It is easy to see that one can convert this initial lower level of surplus into a lower price level and set all the surpluses equal to s_t. Thus, $\mathcal{P}_s$ is not a singleton.

B.4 Proof of Proposition 7

Rollover implies pleasant monetary arithmetic. Suppose that there exists a sequence $\{\tilde{d}_t\}_{t\geq 0}$ such that $\tilde{d}_{t-1} = \phi_t(\tilde{d}_t)\tilde{d}_t$ at any date $t \geq 0$ and $\sum_{t\geq 0} \delta(\tilde{d}_t)$ diverges with $\tilde{d}_{-1} > 0$.

Consider the policy U such that the price level is constant ($P_t = P$), $s_t = 0$ for any date $t \geq 0$ and $P = D_{-1}/\tilde{d}_{-1} > 0$. The path of debt $D_t/P = \tilde{d}_t$ thus satisfies the budget constraint. Besides,

$$\prod_{i=t}^{\tau} \left(\bar{\phi}_i \left(\frac{D_i}{P_{i+1}} \right) \right) \frac{D_\tau}{P_{\tau+1}} = \prod_{i=t}^{\tau} \left(\frac{\bar{\phi}_i}{\phi_i} \right) \frac{D_{-1}}{P_0}. \tag{75}$$

Thus, the log of the product is simply

$$-\sum_{i=t}^{\tau} \delta_i + \ln \left(\frac{D_{-1}}{P_0} \right), \tag{76}$$

that tends to $-\infty$ which proves that the transversality condition is satisfied. Therefore U is feasible.

Let us now prove that there exists a rollover with an initial level of debt $\tilde{d}'_{-1} < \tilde{d}_{-1}$. First, we can construct a sequence $\{\tilde{d}'_t\}_{t\in\mathbb{N}}$ such that $\tilde{d}'_t \leq \tilde{d}_t$, because the functions $d \mapsto d\phi_t(d)$ are increasing and continuous at any date $t \geq 0$. Besides since $\bar{\phi}_t/\phi_t$ is increasing it means that the rollover is associated with a diverging sum of convenience yield. Overall this means that we can construct a feasible policy (P, s) as above but with a higher initial price level and the same stream of surpluses. Monetary arithmetic is thus pleasant.

Pleasant monetary arithmetic implies the existence of a rollover. Suppose there exists a policy $U = \{(P_t, s_t)\}_{t\geq 0}$ that does not feature a tradeoff: There exists another feasible policy U' featuring $P'_0 \leq P_0$ (and the same values of the price levels from date 1 on) and $s'_t \leq s_t$ for any $t \geq 0$ with at least one strict inequality. At date 0,

$$\phi_0\left(d'_0\right)d'_0 = \frac{D_{-1}}{P'_0} - s'_0 \geq \frac{D_{-1}}{P_0} - s_0 = \phi_0\left(d_0\right)d_0$$

Therefore, since the function $d \mapsto d\phi_0(d)$ is strictly increasing, $d'_0 \geq d_0$. The same reasoning leads to $d'_t \geq d_t$ for any $t > 0$ by induction. In addition, there exists (at least) one date $\tau \geq 0$ such that $d_{\tau'} > d_\tau$.

Let us denote for all $i \geq 0$ $q_i = \phi_i(d_i)$, $\bar{q}_i = \bar{\phi}_i(d_i)$, $Q_t = \prod_{i=0}^t q_i$, and $\bar{Q}_t = \prod_{i=0}^t \bar{q}_i$. Comparing the budget constraints for the two policies at any date $t \geq 0$ yields:

$$d'_t - d_t = s'_{t+1} - s_{t+1} + q'_{t+1}(d'_{t+1} - d_{t+1}) + (q'_{t+1} - q_{t+1})d_{t+1}, \tag{77}$$

which implies

$$d'_t - d_t \leq q'_{t+1}(d'_{t+1} - d_{t+1}). \tag{78}$$

Compounding this inequality from $t = \tau$ to an arbitrary $T > \tau$ we find:

$$q'_0 \dots q'_T d'_T \geq q'_0 \dots q'_T d_T + q'_0 \dots q'_\tau (d'_\tau - d_\tau). \tag{79}$$

The first right-hand-side member is positive and the second right-hand-side member is strictly positive and independent of T. Therefore, $Q'_T d'_T$ cannot converge to zero and,

$$Q'_T d'_T = \frac{Q'_T}{\bar{Q}'_T}\bar{Q}'_T d'_T \tag{80}$$

implies that $Q'_T/\bar{Q}'_T$ must be unbounded since $\bar{Q}'_T d'_T \to 0$.

Let now construct a rollover $\{\tilde{d}_t\}$. We define $\tilde{d}_\tau = d'_\tau - d_\tau$. We obtain $\tilde{d}_t$ between 0 and $\tau - 1$ by backward induction using (36). Suppose that we can build a rollover up to date t such that $\tilde{d}_t \leq d'_t - d_t$. Inequality (78) combined with the facts that $(d'_{t+1} - d_{t+1}) < d'_{t+1}$ and that ϕ_t is decreasing show that we have:

$$d'_t - d_t \leq \phi_{t+1}(d'_{t+1} - d_{t+1})(d'_{t+1} - d_{t+1}), \tag{81}$$

which proves that there exists a feasible rollover at date $t+1$, $\tilde{d}_{t+1}$ such that $\tilde{d}_{t+1} \leq (d'_{t+1} - d_{t+1})$.

Finally, the fact that $Q'_T/\bar{Q}'_T$ must be unbounded means also that $\tilde{Q}_T/\bar{\tilde{Q}}_T$ must be unbounded, implying that $\sum_t \tilde{\delta}_t$ diverges.

B.5 Proof of Proposition 8

Let $P' \in \mathcal{P}_{s'}$. If $s'_0 < s_0$, then one can find P such that $(P, s) \in \mathcal{F}$ and $P'_0 > P_0$ by just setting P_0 such that

$$\frac{D_{-1}}{P_0} = s_0 + d'_1 \phi_0(d'_1). \tag{82}$$

We skip for brevity the straightforward construction of the associated debt sequence $(d_t)_{t \geq 2}$.

If $s'_0 = s_0$, then let

$$\tau = \inf\{t \geq 1 \mid s'_t < s_t\}. \tag{83}$$

One can strictly increase $d'_{\tau-1}$ when substituting s'_τ with s_τ and so on until d'_0 given that $d \mapsto d\phi_t(d)$ is strictly increasing, enabling the choice of a $P_0 < P'_0$.

It is easy to see that in addition, $P'_0 - P_0$ is bounded away from 0 over all feasible P', which establishes the result.

B.6 Proof of Proposition 9

Consider a feasible policy (P, s). The existence of a rollover of reserves with diverging $\sum_{t \in \mathbb{N}} \delta^X_t(x_t)$ implies that the central bank can generate positive remittances for instance at date 0 and then rollover the reserves forever. These remittances add up to the revenue of the government and allow for reducing the initial price level or the surpluses at some point.

B.7 Proof of Proposition 10

If reserves and bonds are perfect substitutes (equation (51)), then equation (52) is satisfied and we can apply all the results from Section 4.2 with $D = B - B^B + X$; besides, since θ does not appear in equation (52), the remittances and the central bank's balance sheet are immaterial for the joint determination of (P, s).

More precisely, if the policy without remittances (P, s) is feasible for an initial legacy debt D_{-1}, then there exists a unique path of debt D_t that satisfies the government budget constraint. The decomposition of this debt into (B, X, B^B) is irrelevant, the only relevant quantity being the overall net public liability D_{-1}. Reciprocally, if the extended policy (P, s, θ) is feasible for an initial legacy debt D_{-1} then the policy (P, s) is also feasible for a legacy debt D_{-1}.

As a consequence, we get $\mathcal{P}_s = \mathcal{P}_s^X$ for a given D_{-1}.

In addition, if the monetary arithmetic is unpleasant then the set of feasible prices $\mathcal{P}_s$ for a given stream of surpluses s is simply a singleton—let us call it $\{P_0\}$—and extending the policy space does not affect this set. Therefore, the monetary arithmetic is also unpleasant for the extended policies and the unique feasible price level is determined by the stream of surpluses, that is, $\mathcal{P}_s = \mathcal{P}_s^X = \{P_0\}$.