# A State Theory of Price Levels*

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**Abstract**

This paper studies how public financial policy, defined as the collection of monetary transfers and trades of money between public and private sectors, affects price levels. In an economy in which private agents are free to set the prices at which they privately trade money for goods with each other, we identify policies that elicit a single equilibrium price level. For policies that fail to do so, for example because different official and unofficial prices may coexist in equilibrium, we still offer tight restrictions on the set of predictable price levels.

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# 1 Introduction

The determination of the prices of consumption goods in terms of money is a basic yet largely unsettled question in economics. It is for example telling that the mainstream monetary models that are currently used to inform policy do not determine the price level. Still, public financial policies implement trades and transfers of money with the private sector that presumably play an important role in this determination. First, public sectors—central banks in practice—trade the money that they issue for other stores of value such as foreign currencies in foreign exchange interventions, or (public or private) fixed-income securities in open-market operations. Some of the prices at which such trades settle serve as important nominal anchors, and may thus be explicit targets for monetary policy. Historical and contemporaneous examples abound, including metallic standards, currency pegs, or the targeting of some short-term interest rates. Second, fiscal policy enforces monetary transfers that may also affect the determination of the price level, be it only because money is legal tender for tax liabilities (e.g., Lerner, 1947; Smith, 1776; Starr, 1974). Finally, that public sectors can issue money in order to make good on their own (explicit and implicit) liabilities creates another possible connection between the value of money and the actions of the issuing state (e.g., Sargent and Wallace, 1981).

Does such a collection of transfers and trades of money by the public sector suffice to determine the price level? This paper studies this old question in an economy that has a simple but novel feature: All agents are free to privately trade money for a (single) consumption good with each other at whichever prices they agree upon. Namely, subject to a resource constraint, each agent can submit buy or sell orders of any quantity of goods at any price. Orders at the same price clear with uniform rationing of the long side. This unfettered free trade can be sophisticated in the sense that agents can act as leveraged intermediaries across such endogenously created trading posts with varying prices, bidding in one post the expected proceeds from another one subject to a natural solvency constraint. A public financial policy determines the price level in this economy if all equilibrium trades occur at the same price.

In other words, we study whether the state can determine the price level without resorting to any form of implicit or explicit trading ban nor price control, as is for example *de facto* the case under the assumption of mandatory centralized markets run by a Walrasian auctioneer. We think that this is a natural environment to study robust price-level determination. This generates new insights into the respective contributions of public financial policy *per se* and that of other sometimes less transparent assumptions on social interactions, such as restrictions on trading protocols, to the determination of the price level.

We carry out our analysis in an environment that is fully strategic, and we use Nash equilibrium as our concept of predictable outcome. Unlike with Walrasian environments, this enables a distinction between on one hand the policies that are feasible, and on the other hand the policies that determine the price level among these feasible ones. As we will show, some feasible policies that fail to determine the price level have interesting and plausible properties such as the coexistence of official and unofficial prices. In this case the price level is not determined but all the prices at which money may trade are fully characterized by public financial policy—the collection of trades and transfers between the state and the private sector.

We first study an elementary one-commodity one-date economy. In this economy, public financial policy has three central components: i) a maximum quantity of goods that the state is willing to trade for money, ii) a price at which the state is willing to trade goods for money, and iii) a vector of monetary transfers from the state to each private agent. We deem fiscal debtors the private agents who receive a negative transfer from the state, and fiscal creditors those who receive a positive transfer.

**Fixed policies.** We first study the natural situation in which the price at which the state is willing to trade and the transfers are fixed, as opposed to contingent on the actions of the private sector. This is empirically relevant as a fixed official price resembles a currency peg, a metallic standard, or some of the allocation mechanisms currently used by central banks in their refinancing operations. Fixed transfers correspond to the repayment of nominally safe public liabilities issued in the (for now unmodelled) past such as central-bank reserves.

Our first key insight is that such a combination of fixed transfers and a fixed official trading price determines the price level if and only if it is impossible for any subgroup of private agents to place trades in the official market that would lead other agents to be rationed in it. A policy that does not satisfy this property opens up the possibility of situations in which some agents coordinate on squeezing the official market in order to induce the other agents to trade at unfavorable unofficial prices with them. In order to discourage such behavior, the state must commit to trade quantities of goods or money that are strictly larger than the ones it ends up trading in equilibrium.

More concretely, suppose for example that the large fiscal debtors in this economy cannot purchase enough money from the state to meet their tax liabilities because the other agents buy more money than they need in the official market. This might lead to a situation akin to debt deflation, whereby these large fiscal debtors are willing to offload desirable commodities for cash at a low unofficial price in order to avoid bankruptcy, and their counterparts use the cash that they obtain in the official market to snap up these cheap commodities. The state must ensure that every single debtor can purchase enough cash to afford her taxes at the official price no matter the trades of the rest of the private sector in order to avoid the emergence of such low unofficial price levels. This may imply that the state must stand ready to sell much more money (buy many more goods) than it actually does in equilibrium.

Symmetrically, in the presence of fiscal creditors, some policies may lead to situations of financial repression in which the private sector cannot purchase as many goods from the state as it would like to given its cash holdings and the official price. In this case, the agents with the smallest cash holdings can sell goods to the cash-rich ones at a high unofficial price level, and use the proceeds to improve their bids and increase their share in the official market. Examples of such unofficial markets at inflated prices abound in practice, for example when a currency peg is no longer credible.

In sum, if public financial policy consists in a *fixed* official trading price and given *fixed* transfers, the determination of the price level requires that the quantities that the state stands ready to trade with the private sector be strictly larger than the one that actually changes hands in equilibrium.

**Contingent policies.** Then we study policies that are contingent on the actions of the private sectors in ways that are empirically relevant. Our contingent official price is a market-clearing price à la Shapley and Shubik (1977)—the price that makes the in and out of equilibrium private demand for goods equal to the maximum quantity that the government supplies. Our contingent transfers are defaultable payments, in the sense that the state never transfers more in aggregate than the amount of money that it collects from trading—that is, the state does not create money to honor its liabilities. We find that policies such that the price or/and the transfers are contingent this way never determine the price level. They always create room for trades at unofficial prices. However, if the official price clears the market while transfers are nominally safe, or if, conversely, the official price is fixed while transfers are defaultable, the unofficial prices all converge to a single price level when all private agents become negligible in the sense that the maximum price impact that each of them can have tends to zero.

By contrast, if both the official price and the transfers are contingent—that is, if policy features both a market-clearing price and defaultable transfers, then there is no determination of the price level, not even in this negligible limit. There are multiple equilibria akin to self-fulfilling debt crises whereby private agents demand very few goods in the official market because they expect low payments from the government. This is self-justified, and generates a continuum of equilibria across which price levels decrease with the severity of default.

**Dynamics.** We then write a two-date extension of our model in order to endogenize ex-post transfers as resulting from ex-ante private portfolio decisions. We endogenize a zero lower bound by showing that an excessively low nominal rate on money may lead to the creation of inside money—credit issued by private agents. We also obtain that away from this lower bound, the current price level may still be determined even if agents do not expect the price level to be determined in the future, for example because they expect financial repression. This is so when future equilibrium multiplicity translates into the current indeterminacy of the quantity of money traded rather than on its price.

**Application to other contexts.** Our analysis shows that the actions of the state that turn out to be particularly important, and in particular necessary to determine the price level, are trades of money for desirable commodities. These are not specific to the public sector and can be implemented by any issuer of claims. Thus, our model may also be applied to the determination of the value of a private currency provided one agent has monopoly over its issuance. One only needs to take taxation out of the picture in our setup. Also, the situation of financial repression associated with rationing of private agents in the official trading post is akin to the ones that create room for run equilibria in models following Diamond and Dybvig (1983). In this latter case, money stands for the deposits issued by the bank and the good for its holdings of central bank's reserves.

**Related literature.** The title of this paper is an unsubtle reference to the state theory of money outlined in Knapp (1924). As epitomized by the opening sentence of the book—*"Money is a creature of law."*—the state theory of money contends that the state has a unique ability to impose something as money due to its legislative capacity. Our contribution is to formally study the extent to which this capacity may suffice to determine the price level. Here, the formalization of the distinctive capacity of the state is that it is the only agent which can print money, declare taxes, and expropriate bankrupt private agents.

Bassetto (2002) pioneers the strategic foundations of price-level determination by public financial policy. Its goal is to offer an example of an economy in which the fiscal theory of the price level applies. We share with him a strategically closed environment that highlights the importance of credible out-of-equilibrium actions in shaping equilibrium outcomes. By lifting his restrictions to centralized markets and cash-in-advance, we also generate a number of additional and, we believe, empirically relevant insights.

Given the central role of trade in our game-theoretic framework, we revisit the old and large literature on the strategic foundations of Walrasian equilibrium. A review is beyond the scope of this paper, important contributions include Dubey (1982) and the reference herein, Dubey and Shubik (1980), Schmeidler (1980), and Shapley and Shubik (1977). We show that when an economic good—money—is only valuable to optimizing agents as legal tender, the producer of this good still has the ability to coordinate private agents on a given nominal anchor and to elicit trade.

Finally, the search literature has like us emphasized that the willingness of the state to back its money by accepting to trade it for desirable goods is important (Aiyagari and Wallace, 1997; Li and Wright, 1998). In our frictionless model it is simply a necessary condition for price-level determination. In these papers, this source of value for money coexists with its role of mitigating search frictions, and they show that more backing makes it easier to sustain the Pareto-dominant monetary equilibria.

The paper is organized as follows. Section 2 outlines a static model. Section 3 solves it. Section 4 outlines and solves a two-date model. Section 5 concludes. Proofs follow the propositions because we find them instructive, yet the paper is written so that they can be skipped in a first reading.

# 2 One-date model

This section outlines our simple one-date economy. It presents a baseline public financial policy that consists in fixed negative transfers (taxes), in fixed positive transfers that may be interpreted as extinguishments of nominal liabilities issued in an unmodelled past (reserves with the central bank or nominal bond), and in a commitment to trade a given maximum quantity of goods for money at a fixed official price. Section 3 will solve for the predictable price levels associated with such a baseline policy. It will also compare the outcome with that associated with policies that feature transfers or trading prices that are contingent on the actions of the private sector instead of being fixed as in this baseline one.

## 2.1 Setup

The economy comprises a public sector—“the state”—and $n \geq 2$ private agents indexed over $\mathcal{I} \equiv \{1, ..., n\}$. There are two divisible economic goods, one deemed “the good” and the other “money” henceforth. The good is intrinsically desirable to private agents whereas money is not. Each private agent thus ranks any bundles of the good and money using the standard ordering of their respective quantities of the good only.

The state can produce money at zero cost. Private agents cannot. Each private agent is endowed with $e > 0$ units of the good. All private agents and the state can trade money for the good according to a mechanism described below.

**Public financial policy.** The state enforces a policy that features monetary and in-kind transfers, money creation, and trade. We describe each component of a policy in turn.

**Negative transfers (taxes).** The state levies both in-kind and cash taxes:

- *In-kind taxes.* The state collects a tax of $\tau \in [0, e)$ units of the good on each individual.

- *Taxes paid in cash.* The state requires that each private agent $i \in \mathcal{I}$ pay a tax equal to $T_i \geq 0$ units of money.

To be sure, in-kind taxation is essentially absent in modern economies. One can however more broadly interpret $n\tau$ as the share of the economy's endowment owned by the state.[1]

**Positive transfers.** The state makes a cash transfer $L_i \geq 0$ to each agent $i \in \mathcal{I}$.

**Money creation.** Policy also features the production of $nM \geq 0$ units of money.

**Trade.** The state posts an order to buy a quantity $n\delta_G \in \mathbb{R}$ of the good at the price level $P^* > 0$, with the convention that this is a sell order if $\delta_G \leq 0$.

In sum, a policy consists in a vector $\mathcal{P} = (\tau, (T_i)_{i\in\mathcal{I}}, (L_i)_{i\in\mathcal{I}}, M, P^*, \delta_G)$.

The state also consumes $nc_{G,C} \in \mathbb{R}$ units of the good and $nc_{G,M} \in \mathbb{R}$ units of money. We do not model this consumption as a component of policy but rather as a payoff to the state determined by both policy and by the private sector's strategy profile as detailed below. Bassetto (2002), unlike us, models state spending as a decision that is not contingent on the private sector's strategy, but he posits that taxes, unlike here, are adjusting in response to (in and out of equilibrium) private strategies in order to maintain this fixed spending level. Both approaches are thus equivalent and merely reflect that since the state's surplus depends on voluntary trades by the private sector, then either taxes or expenditures (or both) must be modelled as contingent on actions by all agents—as payoffs rather than actions in a game-theoretic setting.

**Net transfers.** We will make intensive use of the following natural concepts of net transfers associated with a policy $\mathcal{P}$.

**Definition 1.** ***(Net transfers)*** *For all $i \in \mathcal{I}$, let $N_i = L_i - T_i$. Let*

$$N = \frac{1}{n}\sum_{i\in\mathcal{I}} N_i,\ N_+ = \frac{1}{n}\sum_{i\in\mathcal{I}} \max\{N_i, 0\},\ \textit{and}\ N_- = \frac{-1}{n}\sum_{i\in\mathcal{I}} \min\{N_i, 0\}. \tag{1}$$

Notice that $N = N_+ - N_-$. In words, $N$ is the net nominal transfer per capita, $N_+$ is the private net fiscal credit per capita (counting a net debt as zero), and $N_-$ the absolute

[1] We entertain the interpretation of $n\tau$ as in-kind taxation because Section 3.3 shows that our setup can shed light on historical episodes of in-kind taxation by financially distressed states.

value of fiscal net debt per capita (counting a net credit as zero). In the following, we will deem “fiscal creditors” the agents such that $N_i > 0$ and ”fiscal debtors” that for whom $N_i < 0$.

**Private actions.** Taking policy $\mathcal{P}$ as given, private agents play a simultaneous game whereby they make decisions to trade and pay taxes. We describe these decisions in turn, and the resulting payoffs.

**Taxes.** Each private agent $i \in \mathcal{I}$ decides on the amount of cash taxes $\hat{T}_i \geq 0$ that she pays to the state.[2]

**Trades.** Each agent can submit any number of orders to buy or sell a given quantity of goods at a given price. The only restriction is that the total size of her sell orders—the sum of the quantities of goods over all her sell orders—cannot exceed $e - \tau$. This is essentially a no short-sales constraint, as one cannot sell goods that one needs to buy. We will see below that money can by contrast be sold short.

The trading strategy of agent $i \in \mathcal{I}$ is conveniently described by the functions describing her cumulative orders. The respective cumulative buy and sell orders at prices (weakly) lower than $P$, $D_i(P)$ and $S_i(P)$ respectively, are increasing step functions over $[0, +\infty)$ satisfying:

$$D_i(0) = S_i(0) = 0, \tag{2}$$

$$\lim_{+\infty} S_i \leq e - \tau. \tag{3}$$

Let us denote for all $P > 0$ $d_i(P)$ and $s_i(P)$ the respective quantities of the good that $i$ respectively seeks to buy and sell at the price $P$:

$$d_i(P) \equiv \int \mathbb{1}_{\{p=P\}} dD_i(p), \;\; s_i(P) \equiv \int \mathbb{1}_{\{p=P\}} dS_i(p). \tag{4}$$

In sum, the strategy of agent $i \in \mathcal{I}$ is $\mathcal{S}_i = (\hat{T}_i, D_i(.), S_i(.))$. Let $\mathcal{S} = (\mathcal{S}_i)_{i \in \mathcal{I}}$ denote the strategy profile of the private sector.

[2]Notice that unlike the cash taxes $T_i$, the in-kind taxes $\tau$ are automatically collected. This is consistent with the interpretation that $n\tau$ is just the state's real endowment, and will simplify the bankruptcy process.

**Market clearing and bankruptcy mechanism.** We now describe how market clearing and a bankruptcy mechanism shape the payoff of each agent given a policy $\mathcal{P}$ and a strategy profile $\mathcal{S}$.

**Market clearing.** For all $P > 0$, let $d(P)$ and $s(P)$ denote the aggregate buy and sell orders at the trading post $P$:

$$d(P) = \sum_{i \in \mathcal{I}} d_i(P) + \mathbb{1}_{\{P=P^*\}} \delta_G^+, \;\; s(P) = \sum_{i \in \mathcal{I}}^{n} s_i(P) + \mathbb{1}_{\{P=P^*\}} (-\delta_G)^+ \tag{5}$$

If $d(P)s(P) = 0$, then no trade takes place. Otherwise, the smallest side of the market is fully executed and the other side is rationed pro rata the size of each order. Formally, each private agent $i \in \mathcal{I}$ buys and sells effective quantities $\hat{d}_i(P)$ and $\hat{s}_i(P)$ such that:

$$\hat{d}_i(P) \equiv d_i(P) \min\left\{1, \frac{s(P)}{d(P)}\right\}, \;\; \hat{s}_i(P) \equiv s_i(P) \min\left\{1, \frac{d(P)}{s(P)}\right\}, \tag{6}$$

and the same uniform rationing rule applies to the state at $P = P^*$. We respectively denote $\hat{D}_i(P)$ and $\hat{S}_i(P)$ the respective cumulative effective purchases and sales of agent $i \in \mathcal{I}$.

The following definition is natural and important. It states that a trading post is active if and only if at least one private agent strictly gains or loses goods in it.

**Definition 2.** ***(Active trading post, net buyer, net seller)*** *Trader $i \in \mathcal{I}$ is net buyer (respectively net seller) at the trading post $P$ if and only if $\hat{d}_i(P) > \hat{s}_i(P)$ ($\hat{s}_i(P) > \hat{d}_i(P)$ respectively). The trading post $P$ is active if and only if at least one private agent is net buyer or net seller at $P$.*

An active trading post always features both at least one net buyer and one net seller by definition, but one of them can be the state.

**Why uniform rationing?** The only property of uniform rationing that is crucial for our results is that a larger bid generates other things being equal a larger allocation. Many other trading games share this property, including for example the sequential clearing of bids at the same price in random order.[3]

[3] This example would require explicit assumptions on how private agents assess random consumption bundles.

**Bankruptcy mechanism.** Suppose that agent $i \in \mathcal{I}$

- either places orders such that

$$\int PdD_i(P) > L_i - \hat{T}_i + \int Pd\hat{S}_i(P), \tag{7}$$

- or pays taxes $\hat{T}_i < T_i$,

then the state seizes all the goods and money of that agent and replaces her in the market. The state creates all the money that is needed to execute this agent's buy orders and/or to make up for $T_i - \hat{T}_i$.

In words, if an agent either places buy orders for a total amount larger than $L_i - \hat{T}_i$ plus the cash value of her *effective* sell orders or/and defaults on her taxes, then she is bankrupt and does not consume anything.

In this Bertrand-Cournot market structure, unlike in Shapley-Shubik games, there is no cash-in-advance constraint: An agent can pledge in a post the cash proceeds from the simultaneous sales of her goods on another post. The bankruptcy rule means however that the agent must "mark to market" when doing so: She cannot make up arbitrary valuations of her marketed goods. Betting more cash than the effective proceeds from selling her goods is punished by bankruptcy.

This bankruptcy rule borrows from Dubey (1982). It plays two distinct roles. First, it ensures that there is no default contagion in the sense that an agent does not have to worry that her buying decisions may trigger a chain of defaults affecting her counterparties' orders. This is because the state steps in and makes good on the commitment of her counterparts if they fall short of cash because of her. Notice that if this was the sole purpose of the bankruptcy rule, it would be sufficient to posit that each agent can honor its effective orders and tax payments to avoid bankruptcy. Namely, one could write $\int Pd\hat{D}_i(P)$ on the left-hand side of the bankruptcy condition (7), which is (weakly) smaller than $\int PdD_i(P)$.

Imposing that the whole buy orders of an agent, $\int PdD_i(P)$, rather than only the effective ones matter to trigger bankruptcy plays a second role. It creates a cost for each agent to dilute the others with arbitrarily large buy orders that will be only partially executed. To dilute other buyers this way, an agent must have enough collateral in the form of sufficient cash obtained either through transfers $N_i$ (possibly negative) and effective sales. Such a natural collateral constraint is important in our setup in which, unlike in Dubey (1982), rationing may occur in equilibrium.[4]

**An illustrative example.** To see concretely how trading limits work, suppose that three agents $A$, $B$, and $C$ trade. $B$ sells one unit at a unit price and one unit at a price of 2. $C$ buys 0.5 units at 2 and one unit at one. $A$ sells one unit at 2. When the market at 2 clears, there is excess supply and $A$ receives 0.5 units of cash from this market effectively selling only 0.25 units. $A$ is bankrupt if she bids to buy more than $L_A - \hat{T}_A + 0.5$ units in the price-1 market.

**Payoffs.** The payoffs associated with $\mathcal{P}$ and $\mathcal{S}$ result immediately from this trading structure and bankruptcy rule. An agent $i \in \mathcal{I}$ does not consume goods nor money if bankrupt, and consumes otherwise respective quantities of goods and money $c_{i,C}$ and $c_{i,M}$:

$$c_{i,C} = e - \tau + \int d\hat{D}_i(P) - \int d\hat{S}_i(P), \tag{8}$$

$$c_{i,M} = L_i - \hat{T}_i + \int Pd\hat{S}_i(P) - \int Pd\hat{D}_i(P). \tag{9}$$

Notice that these consumptions are positive by construction. The consumption of goods and money by the state in the absence of private bankruptcy, $c_{G,C}$ and $c_{G,M}$, are by conservation of quantities:

$$nc_{G,C} = ne - \sum_{i \in \mathcal{I}} c_{i,C}, \tag{10}$$

$$nc_{G,M} = nM - \sum_{i \in \mathcal{I}} c_{i,M}. \tag{11}$$

**Remark on positive state consumptions.** These state consumptions are not necessarily positive. Section 3.4 will introduce restrictions on policies such that the state consumes positively no matter the private strategy profile. We will deem policies that satisfy these restrictions "feasible". Our main results however hold for any policy, whether

[4] It is possible to show that agents would spontaneously impose on themselves this constraint (7) with whole buy orders in an extension in which policy is uncertain and the state fully accommodates buy orders at $P^*$ with some probability. Private agents would then dilute each other in this $P^*$-post using only cash that they effectively obtain in other posts in order to avoid bankruptcy risk if the state fully meets demand.

they are “feasible” in this sense or not, and so we will present them abstracting from these restrictions.

## 2.2 Some definitions

We model social interactions as a game between private agents given policy. It is thus natural to adopt Nash equilibrium as our concept of predictable outcome:

**Definition 3.** ***(Predictable outcomes)*** *The predictable outcomes of this economy given policy $\mathcal{P}$ are the Nash equilibria played by private agents. Formally, a predictable outcome is a strategy profile $\mathcal{S}$ such that for every $i \in \mathcal{I}$, $\mathcal{S}_i$ maximizes $c_{i,C}$ given $\mathcal{S}_{-i}$ and $\mathcal{P}$.*

Recall that private agents do not care about their consumption of money $c_{i,M}$. This definition of predictable outcomes yields a natural definition of predictable price levels:

**Definition 4.** ***(Predictable price levels)*** *A price $P > 0$ is predictable given policy $\mathcal{P}$ if and only if there exists a predictable outcome (i.e., a Nash equilibrium) with active trading at $P$. Let $\Pi(\mathcal{P})$ denote the set of predictable price levels associated with a policy $\mathcal{P}$.*

This enables us in turn to characterize whether a public financial policy determines the price level:

**Definition 5.** ***(Determination of the price level)*** *A policy weakly determines the price level if and only if $\Pi(\mathcal{P})$ is a singleton. A policy strongly determines the price level if and only if it weakly determines the price level and every Nash equilibrium features active trade.*

The price level may fail to be determined for three reasons. First, it may be that there exists no equilibrium with active trade. Second, it may be that every equilibrium features active trade at a given equilibrium price, but that this latter price varies across equilibria. Finally, an equilibrium may feature active trades at different prices. We will see that there exist policies leading to each of these three configurations, together with the ones that actually determine the price level.

# 3 Analysis

We solve for the predictable outcomes of this economy given a policy $\mathcal{P}$ with a focus on the determination of the price level. The following section first introduces important properties of equilibrium trades.

## 3.1 Some properties of equilibrium trades

The following lemma first shows that one can without loss of generality offset trades by the same agent at a given price level.

**Lemma 1.** ***(Netting)*** *Consider a strategy profile such that agent $i \in \mathcal{I}$ is a non-bankrupt net buyer at the trading post $P$. If she deviates and sets $s_i'(P) = 0$, $d_i'(P) = d_i(P) - d(P)s_i(P)/s(P)$ then she does not affect her allocation nor that of other agents. Symmetrically, suppose she is net seller at $P$. If she deviates and sets $d_i'(P) = 0$, $s_i'(P) = s_i(P) - s(P)d_i(P)/d(P)$ then she does not affect her allocation nor that of other agents.*

*Proof.* The results stem directly from the fact that these deviations do not affect the rationing coefficients as when $s(P) \neq s_i(P)$ and $d(P) \neq d_i(P)$,

$$\frac{d(P) - \frac{d(P)si(P)}{s(P)}}{s(P) - s_i(P)} = \frac{d(P) - d_i(P)}{s(P) - \frac{s(P)di(P)}{d(P)}} = \frac{d(P)}{s(P)}.$$

The effective trades of the other agents are thus unaffected by the deviation of $i$. Nor is the post-trade allocation for $i$. To see this, suppose that $i$ is net buyer. The reduction in her buy order $d(P)s_i(P)/s(P)$ is weakly larger than that of her effective sales $s_i(P)\min\{d(P)/s(P), 1\}$, and so the deviation leaves her solvent. Furthermore,

$$\hat{d}_i'(P) - \hat{s}_i'(P) = \left(d_i(P) - d(P)\frac{s_i(P)}{s(P)}\right)\min\left\{1, \frac{s(P)}{d(P)}\right\} = \hat{d}_i(P) - \hat{s}_i(P),$$

leaving her allocation unchanged. The same reasoning applies for a net seller. □

This result is useful because it implies that whenever an agent is net seller or net buyer at one post, we can assume that she nets her trades this way before entering into a deviation so that we do not have to worry about the impact of small deviations from her larger effective order on her potential order on the other side.

For any active trading post $P$ and any $i \in \mathcal{I}$, let us now define:

$$\Delta_i(P) = \begin{cases} \frac{s(P)(d(P)-d_i(P))}{d(P)^2} \text{ if } s(P) \leq d(P), \\ 1 \text{ otherwise.} \end{cases} \tag{12}$$

The coefficient $\Delta_i(P)$ measures the marginal return from increasing a buy order in a market in which buyers are (weakly) rationed ($s(P) \leq d(P)$). On one hand a marginal increase $\epsilon$ in an order generates $\epsilon s(P)/d(P)$ additional marginal units. On the other hand it crowds out the outstanding order $d_i(P)$ thereby costing a marginal reduction $\epsilon(d_i(P)/d(P)) \times (s(P)/d(P))$ in the return on this outstanding order.

The following two lemmas introduce some important characteristics of equilibrium trades at multiple prices.

**Lemma 2.** ***(Selling high to buy low)*** *Suppose that in an equilibrium that features (at least) two active trading posts with price levels $P$ and $P'$, a non-bankrupt agent $i \in \mathcal{I}$ is net seller at $P'$ and net buyer at $P$. Then $P' > P$, and if $s(P) < d(P)$,*

$$P'\Delta_i(P) \geq P. \tag{13}$$

*Proof.* Let us define:

$$\delta(P, \epsilon) = \min\left\{1, \frac{s(P)}{d(P)+\epsilon}\right\} \text{ and } \sigma(P, \epsilon) = \min\left\{1, \frac{d(P)}{s(P)+\epsilon}\right\}.$$

We let $i$ modify her orders as follows. She first nets her positions as in Lemma 1. If she is the only seller at $P'$ and is rationed, she also reduces her order up to the total buy order. For notational simplicity we maintain the notation $s_i(P')$, $d_i(P)$ for this new orders.

From this position agent $i$ reduces her sell order at $P'$ by $\gamma\epsilon$ and her buy order at $P$ by $\eta(\epsilon) = \epsilon P'\sigma(P', -\gamma\epsilon)/P$, where $\epsilon > 0$ is sufficiently small and $\gamma > 0$ is defined as follows. If $d(P') \geq s'(P')$, $\gamma = 1$, and $\gamma \geq s(P')/(s(P') - s_i(P'))$ otherwise. Notice that this leaves the agent solvent since her buy order and her effective sell order shrink by the same amount. The net change in consumption units resulting from this deviation is

$$
\begin{aligned}
&(d_i(P) - \eta(\epsilon))\delta(P, -\eta(\epsilon)) - d_i(P)\delta(P, 0) - (s_i(P') - \gamma\epsilon)\sigma(P', -\gamma\epsilon) + s_i(P')\sigma(P', 0) \\
=&d_i(P)(\delta(P, -\eta(\epsilon)) - \delta(P, 0)) + \eta(\epsilon)\left(\frac{P}{P'} - \delta(P, -\eta(\epsilon))\right) + s_i(P')(\sigma(P', 0) - \sigma(P', -\gamma\epsilon)) \\
+&(\gamma - 1)\epsilon\sigma(P', -\gamma\epsilon).
\end{aligned}
$$

At first-order in $\epsilon$, the first two terms are equal to

$$
\epsilon\frac{d(P)P'}{s(P)P}\left(\frac{P}{P'} - \delta(P) + \mathbb{1}_{\{\delta(P)<1\}}\delta(P)\frac{d_i(P)}{d(P)}\right),
$$

and the last two terms are of second order in $\epsilon$. Thus this deviation yields a strict benefit if $P' < P$ or if $\delta(P) < 1$ and (13) does not hold, a contradiction. □

Intuitively, selling high to buy low is profitable only if the redeployment of the sales proceeds to buy in the cheap trading post does not crowd out the outstanding order at this post. The following lemma offers a necessary condition for a private agent being willing to buy at the highest of two prices.

**Lemma 3.** ***(Buying high instead of low)*** *Suppose that an equilibrium features (at least) two active trading posts with price levels $P$ and $P' > P$. If a non-bankrupt agent $i \in \mathcal{I}$ is net buyer at $P'$ then*

$$
P'\Delta_i(P) \leq P\Delta_i(P'). \tag{14}
$$

*Proof.* Notice that Lemma 2 ensures that $i$ cannot be a net seller at $P$ as $P' > P$. We let $i$ modify her orders as follows. She first nets her positions as in Lemma 1. Then she reduces her buy order at $P'$ by $\epsilon$ and increases her buy order at $P$ (possibly equal to 0) by $\epsilon P'/P$, where $\epsilon > 0$ is sufficiently small. Her solvency constraint still holds since the total cash value of her buy orders is unchanged. The net change in consumption units resulting from this deviation is

$$
\begin{aligned}
&(d_i(P') - \epsilon)\delta(P', -\epsilon) - d_i(P')\delta(P') + \left(d_i(P) + \epsilon\frac{P'}{P}\right)\delta\left(P, \epsilon\frac{P'}{P}\right) - d_i(P)\delta(P) \\
=&\epsilon\left(\frac{P'}{P}\delta\left(P, \epsilon\frac{P'}{P}\right) - \delta(P', -\epsilon)\right) + d_i(P)\left(\delta\left(P, \epsilon\frac{P'}{P}\right) - \delta(P)\right) \\
&+ d_i(P')\left(\delta\left(P', -\epsilon\right) - \delta(P')\right).
\end{aligned}
$$

As $\epsilon \downarrow 0$, this is equivalent to

$$
\epsilon\frac{P'\delta(P)}{P}\left[1 - \mathbb{1}_{\{\Delta_i(P)<1\}}\frac{d_i(P)}{d(P)}\right] - \epsilon\delta(P')\left[1 - \mathbb{1}_{\{\Delta_i(P')<1\}}\frac{d_i(P')}{d(P')}\right],
$$

strictly positive if condition (14) does not hold, which establishes the result. □

Intuitively, an agent is willing to be net buyer at $P' > P$ if her order at $P$ is sufficiently large that she would crowd herself out by rebalancing some of her expensive order $P'$ towards $P$.

## 3.2 Some necessary conditions for price-level determination

Next, we identify two properties of policies that each entail that the policy fails to determine the price level. First, and not surprisingly, the absence of net transfers implies indetermination of the price level because there exists no equilibrium with active trading in this case.

**Lemma 4.** ***(No price-level determination without transfers)*** *If a policy is such that* $N_+ + N_- = 0$ *there is no determination of the price level because there is no equilibrium with active trading:* $\Pi(\mathcal{P}) = \emptyset$.

*Proof.* Suppose that there exists at least one active trading post. If all private agents are all net buyers on every active post, then they are all bankrupt, a contradiction since they would be strictly better off not trading. Thus one at least is net seller. Let $\underline{P}$ denote the smallest price at which there is a private net seller. She must be net buyer somewhere else otherwise she would be strictly better off not trading. From Lemma 2, it has to be at a lower price, but since $\underline{P}$ is the smallest price at which there is a private net seller, the net seller must be the governement so $P^* < \underline{P}$. But then this means there is a private net buyer at $\underline{P}$, as it cannot be the state which buys at this price. Let $\bar{P} \geq \underline{P}$

denote the largest price at which there is a private net buyer. A net buyer at this price cannot be net seller at any lower price from Lemma 2. But then she must be bankrupt, a contradiction. □

Second, and more interestingly, we show that any policy that creates gains from trade between private agents because some are fiscal debtors and other fiscal creditors fails to determine the price level.

**Lemma 5.** ***(Private gains from trade preclude the determination of the price level)*** *If a policy is such that $N_+N_- > 0$ then it does not determine the price level because $\{P^*\} \subsetneq \Pi(\mathcal{P})$.*

*Proof.* Without loss of generality, we suppose that $(N_i)_{i\in\mathcal{I}}$ is increasing. We denote $n_-$ the (strict) fiscal debtor with the smallest debt (the smallest absolute value of $N_i < 0$). Notice first that there always exists an equilibrium with a single trading post at $P^*$ in which agent $i \in \mathcal{I}$ submits a buy order $N_i/P^*$ if $i > n_-$ and sells $\min\{e-\tau, -N_i/P^*\}$ otherwise. This "$P^*$-equilibrium" features no bankruptcy if and only if $P^*(e-\tau)+N_i \geq 0$ for all $i \in \{1, ..., n_-\}$ and $N + P^*\delta_G \geq 0$.

We construct another equilibrium in which there is active trade at two prices, $P^*$ and $P > P^*$. We first construct the equilibrium supposing that $N_n > N_{n-1}$, and then will extend it to the case in which several agents share this same highest value of net transfers $N_n$.

**Step 1.** Suppose first that the $P^*$-equilibrium features no bankruptcy. We construct an equilibrium in which agent $n$ invests a sufficiently small nominal amount $B$ in a trading post $P > P^*$. All the other agents are on the other side of the market at $P$. The other fiscal creditors (if any) redeploy the proceeds from selling their entire net endowment $e - \tau$ at $P$ in the $P^*$-post. The fiscal debtors mix sales at $P^*$ and at the rationed higher price $P$ so as to meet their liabilities at the lowest cost.

Formally, for $B > 0$ sufficiently small, define $S(B)$ the positive solution to

$$\frac{n_-(e-\tau) - S(B)}{(n-1)(e-\tau) - S(B)} B + P^* S(B) = nN_-. \tag{15}$$

For $B$ sufficiently small, for every $i \in \{1, ..., n_-\}$, there exists a strictly positive solution

$s_i(P^*)$ to

$$\frac{(e-\tau)-s_i(P^*)}{(n-1)(e-\tau)-S(B)}B+P^*s_i(P^*)=-N_i, \tag{16}$$

and by definition

$$\sum_{i=1}^{n_-} s_i(P^*)=S(B). \tag{17}$$

The equilibrium is then as follows. Fiscal debtor $i \in \{1,...,n_-\}$ sells $s_i(P^*)$ at the $P^*$-post and $e-\tau-s_i(P^*)$ at the $P$-post where $P$ is defined below. Agent $j \in \{n_-+1,...,n-1\}$ (if any) sells $e-\tau$ at $P$ and invests a nominal amount equal to the proceeds plus $N_j$ at $P^*$. Agent $n$ invests a nominal amount $N_n - B$ at $P^*$ and $B$ at $P$. The supply at $P^*$ is thus $s(P^*) = n(-\delta_G)^+ + S(B)$, the demand $d(P^*) = n\delta_G^+ + nN_+/P^* - B(n_-(e-\tau) - S(B))/[P^*[(n-1)(e-\tau)-S(B)]] = n\delta_G^+ + nN_+/P^* - nN_-/P^* + S(B)) \geq s(P^*)$. Let us define

$$P=\frac{P^*d(P^*)^2}{s(P^*)\left(d(P^*)-\frac{N_n-B}{P^*}\right)}. \tag{18}$$

Suppose $B$ is sufficiently small that $N_n - B > N_{n-1} + B(e-\tau)/[(n-1)(e-\tau)-S(B)]$. Then $n$'s trade is optimal from (18) and Lemma 3. So are the trades of the other fiscal creditors because Lemma 2 and (18) imply that they would like to sell more at $P$ to reinvest at $P^*$ but they hit their maximum supply $e-\tau$ at $P$. Finally, fiscal debtors cannot meet their net liabilities at a lower cost as they sell as much as possible at $P > P^*$ subject to being solvent.

**Step 2.** Suppose now that the $P^*$-equilibrium features at least one bankrupt agent because there exists $i \in \{1,...,n_-\}$ such that $P^*(e-\tau) < -N_i$ or because $N + P^*\delta_G < 0$. We re-create essentially the same equilibrium as in Step 1. First, for any fiscal debtor $i \leq n_-$ such that $P^*(e-\tau) < -N_i$, replace $-N_i$ with $P^*(e-\tau)$. Second, take one bankrupt agent, and make him add a buy order larger than $N_n/P^*$ (which of course will be executed by the state) at $P^*$ such that overall $N' + P^*\delta_G \leq 0$ where the new aggregate transfer per capita $N'$ factors in the revised sell and buy orders of the bankrupt agents. It is easy to see that replacing this buy order of the bankrupt agent by another one split between $P^*$ and $P$ defined as in Step 1 for $B$ sufficiently small is an equilibrium.

**Step 3.** It is straightforward to adapt the proof to the case in which $k > 1$ agents share the same maximum transfer $N_n$ by letting each of them invest a nominal amount $B/k$ in a $P$-post defined as in Step 1.

**Remark on other equilibria.** Given that the goal of the proof is to offer one example of indetermination, we focused on a particular equilibrium with multiple active prices that has the advantage of being sustainable for any policy such that $N_- N_+ > 0$. To be sure, there are in general plethora of other equilibria, including some with active trade at lower prices than $P^*$. Characterizing them further is not in the scope of this paper. As an illustration of this multiplicity, it is easy to see that in the case in which $\delta_G = 0$, there exists an equilibrium with a single active post with price $P$ for any $P > 0$. □

Lemma 5 states that if a policy opens up potential gains from trade between private agents because the transfers create both fiscal creditors and fiscal debtors, then it cannot determine the price level. Accordingly, in the remainder of the analysis, we will first focus on policies such that all net transfers have the same sign ($N_+ N_- = 0$). We will then show that the insights that we obtain in such situations extend naturally to policies that create both fiscal creditors and debtors provided one extends our baseline policies to ones with two official trading posts in opposite directions.

The essential reason private gains from trade make it impossible to peg the value of money with a single trade is that money serves no other purpose than dodging bankruptcy in this economy.[5] Thus fiscal creditors are happy to trade money for goods at any price. Symmetrically, debtors are happy to trade goods for money at any price provided this makes them solvent. (They also are indifferent between any trade in the absence of any way out of bankruptcy.) The single trade of the state is thus not sufficient to coordinate the private sector on its price-level target $P^*$. In the presence of gains from trade between them, private agents can always simultaneously trade on this official market and on unofficial ones at different price levels. Again, we will see in Section 3.3 that in this case in which $N_- N_+ > 0$, the state can essentially determine the price level with two trades in opposite directions.

[5] Section 4 develops a two-date version of the model in which money may also be desirable as a store of value.

## 3.3 Characterization of price-level determination

Having eliminated feasible policies that fail to determine the price level, we now characterize the ones that do. Lemmas 4 and 5 imply that a policy that determines the price level must be such that $N_+ + N_- > 0$ and $N_+ N_- = 0$. In words, there must be net transfers and they must all have the same sign. Consider first the case in which there are only fiscal debtors.

**Proposition 6.** ***(Fiscal debtors and debt deflation)*** *Suppose that a policy $\mathcal{P}$ is such that $N_- > 0$ and $N_+ = 0$. There is strong price-level determination if and only if*

$$N_i + P^* \min\{\delta_G, e - \tau\} \geq 0 \text{ for all } i \in \mathcal{I}, \tag{19}$$

*in which case $\Pi(\mathcal{P}) = \{P^*\}$. There exist equilibria without bankruptcy if and only if*

$$N_i + P^*(e - \tau) \geq 0 \text{ for all } i \in \mathcal{I} \text{ and } N + P^* \delta_G \geq 0. \tag{20}$$

*In any equilibrium without bankruptcy, active prices are in $(0, P^*]$.*

*Proof.* Suppose first that an equilibrium is without bankruptcy. This implies that there must be active trading. Suppose that the highest active-trading price is strictly above $P^*$. Any net buyer at this price is a private agent and does not sell anywhere from Lemma 2. But then she must be bankrupt, a contradiction.

Notice then that there always exists an equilibrium with a single trading post at $P^*$ in which agent $i \in \mathcal{I}$ sells $\min\{e - \tau; -N_i/P^*\}$. This "$P^*$-equilibrium" features no bankruptcy if and only if condition (20) holds. If it does not hold, any equilibrium without bankruptcy would require that the agents that are bankrupt in the $P^*$-equilibrium can sell goods at a strictly higher price than $P^*$, a contradiction from the above point.

Finally, we show that this $P^*$-equilibrium is the only equilibrium if and only if condition (19) holds. Suppose first that it holds. There is no equilibrium without active trading otherwise any agent such that $N_i < 0$ would be better off deviating and escaping bankruptcy by selling $-N_i/P^*$ at $P^*$. In any equilibrium in which there is trade at another price than $P^*$, there has to be a private net buyer and a private net seller. Let $\underline{P}$ denote the lowest price at which there is a private net seller $i$ and $\bar{P}$ denote the highest price at which there is a private net buyer $j$. Net buyer $j$ cannot sell at any lower price than $\bar{P}$ from Lemma 2 but must sell somewhere to avoid bankruptcy, which she could always achieve from condition (19). Thus she must sell at $P^* > \bar{P}$. But then $i$, who is not net buyer at any post from condition (13) and $P^* > \underline{P}$, would be strictly better off selling only at $P^*$ the amount required to pay her taxes, a contradiction. Condition (19) warrants that she is never too diluted by the other orders to achieve this.

Suppose now that condition (19) does not hold. Suppose first that at least one agent is bankrupt in the $P^*$-equilibrium. Then this agent can add to her equilibrium sale large purchase orders split over $P^*$ and another price $P > P^*$ as in the proof of Proposition 1. Suppose then that no agent is bankrupt in the $P^*$-equilibrium—that is, that condition (20) holds. We construct another equilibrium than the $P^*$-equilibrium as follows. Suppose that all agents sell their entire endowment $e - \tau$ at $P^*$. At least one agent must be bankrupt since (19) fails to hold. At least one agent is strictly not bankrupt since (20) holds. Suppose that bankrupt agents sell an amount $\eta > 0$ of their endowment at $0 < \epsilon < P^*$ instead of $P^*$, and that strictly non-bankrupt agents bet their excess cash from the $P^*$ post at $\epsilon$. From Lemma 2 this is an equilibrium if $\epsilon$ is taken sufficiently small.

**Debt-deflation equilibria without bankruptcy.** Whereas this equilibrium whereby some agents force the bankruptcy of the most indebted ones is simple, we can show that at least for some parameter values, there also exist equilibria in which the least indebted agents force the most indebted ones to shed goods at a low price, yet the unofficial price is sufficiently high that the latter agents can still dodge bankruptcy. □

To grasp the intuition for the results behind Proposition 6, it is useful to start with the remark that there always exists an equilibrium with a single trading post at $P^*$, in which agent $i \in \mathcal{I}$ sells $\min\{e - \tau; -N_i/P^*\}$. We deem this equilibrium the "$P^*$-equilibrium".

If condition (20) fails to hold, then at least one agent must be bankrupt in this $P^*$-equilibrium, either because one agent does not have enough goods to sell to pay her net taxes ($-N_i > P^*(e - \tau)$) or because their total demand for money $-N$ is larger than what the state is willing to supply $P^*\delta_G$. In this case there is no price-level determination because this bankrupt agent is indifferent between this situation and others in which, for example, she is also bankrupt because she supplies money that she does not have to the other agents at a price $P > P^*$.

More interestingly, if (20) holds, all agents are solvent in this $P^*$-equilibrium. Yet, if there is sufficient heterogeneity across fiscal debts, that is, if condition (19) is not satisfied, one can construct debt-deflation equilibria. In these equilibria, there is a stampede in the official market for cash: All the agents sell many goods at $P^*$ and so everybody is rationed in cash. The largest fiscal debtors then cannot collect enough cash in the official market to afford their taxes. They are willing to shed their goods at much lower prices whereas the agents with low fiscal debt recycle the excess cash that they get on the official market to snap up these cheap goods. In other words, agents coordinate on an excessive demand of money in the official market, which creates financial distress for the largest debtors and deflation that benefits the smallest ones.[6]

It is worthwhile stressing that the state can eliminate such debt-deflation equilibria by merely increasing $\delta_G$, or committing to buy more goods thereby injecting enough money in the economy so that condition (19) holds. If for some reason it is unwilling to do so, it must hope for the private sector to coordinate on the $P^*$-equilibrium instead of one with debt deflation.

**"Whatever it takes."** Notice that condition (19) ensuring price-level determination entails that as soon as debtors are heterogeneous, the state must commit to buy strictly more goods than it will in equilibrium in order to eliminate debt-deflation equilibria. The state must commit to do whatever it takes to ensure that each single fiscal debtor can purchase money to honor her liabilities regardless of the actions of the rest of the private sector.

Consider now the situation in which there are only fiscal creditors.

**Proposition 7.** ***(Fiscal creditors and financial repression)*** *Suppose that a policy $\mathcal{P}$ is such that $N_+ > 0$ and $N_- = 0$. There are three types of predictable outcomes:*

1. ***No active trade.*** *If $\delta_G \geq 0$, then $\Pi(\mathcal{P}) = \emptyset$.*

2. ***Strong price-level determination.*** *If $N + P^*\delta_G < 0$, then $\Pi(\mathcal{P}) = \{P^*\}$ and the policy strongly determines the price level. Furthermore,*

$$c_{G,C} = \tau - \frac{N}{P^*}, \ c_{G,M} = M. \tag{21}$$

[6] Without ambiguity, deflation or inflation refers in this static context to the difference between equilibrium and official prices.

3. ***Financial repression.*** *If* $0 < -P^*\delta_G \leq N$*, then there is strong determination of the price level iff* $N_i = N > -P^*\delta_G$ *for all* $i \in \mathcal{I}$*. Otherwise, there also exist equilibria with multiple active trading posts, with all unofficial prices strictly above* $N/(-\delta_G)$*. Whether the price level is determined or not, it is always the case that:*

$$c_{G,C} = \tau + \delta_G \geq \tau - \frac{N}{P^*}, \tag{22}$$

$$c_{G,M} = M - N - P^*\delta_G \leq M. \tag{23}$$

*Proof.* Notice first that fiscal creditors can always avoid bankruptcy by not trading, and find it strictly preferable to going broke.

We show that $\Pi(\mathcal{P}) = \emptyset$ when $\delta_G \geq 0$. Suppose otherwise that there is an active trading post. There has to be an active private net seller since the state buys. At the lowest price at which there is a private net seller, this net seller does not buy at a higher price from Lemma 2 and cannot buy lower. She would thus be strictly better off reducing her order, a contradiction.

Suppose then that $\delta_G < 0$. There is no equilibrium with no trade as one agent could deviate and buy goods from the state. It is easy to see that there exists a "$P^*$-equilibrium" in which each private agent $i \in \mathcal{I}$ places a buy order for $N_i/P^*$ units at $P^*$. Suppose there exists an equilibrium with active trading at another price. There has to be active net sellers and buyers at this other price. Let us denote $\underline{P}$ the lowest price at which there is an active net seller $i$ and $\bar{P}$ the highest price at which there is an active net buyer $j$. Agent $i$ must be buying too, below $\underline{P}$ from Lemma 2, and by definition cannot do so from a private seller, so it does so at $P^* < \underline{P}$.

Suppose $N + P^*\delta_G < 0$. In this case buyers at $P^*$ cannot be rationed since the private sector as a whole cannot bid more than $N$ at $P^*$ in an equilibrium without bankruptcy. Condition (14) implies that there cannot be a private net buyer at $\underline{P} > P^*$. This shows that the $P^*$-equilibrium is unique.

Suppose $N + P^*\delta_G \geq 0$. Condition (13) implies that $\underline{P} > N/(-\delta_G)$. Suppose that $N + P^*\delta_G > 0$ and $N_i = N$ for all $i \in \mathcal{I}$. Conditions (13) and (14) together imply $\Delta_i(P^*) \geq \Delta_j(P^*)$, requiring that $i$ is net buyer above $\underline{P}$ or/and $j$ is net seller below $\bar{P}$, either way a contradiction.

It only remains to construct an equilibrium with unofficial trade when $N + P^*\delta_G = 0$

or $N + P^*\delta_G > 0$ and net transfers are heterogeneous. It is easy to see that the type of equilibrium that we construct in the proof of Lemma 4 also holds in the absence of fiscal debtors. The only case which slightly differs is that in which $N + P^*\delta_G = 0$ and all agents are ex ante identical. In this case, one equilibrium can be such that one of them buys at an unofficial price defined as in the proof of Lemma 4. The others sell all their goods at this price and reinvest the proceeds in the official market. Unlike when $N + P^*\delta_G > 0$ and agents are ex ante identical, this is an equilibrium as condition (13) is not necessary in the case in which $N + P^*\delta_G = 0$. □

Proposition 7 first states that in the presence of heterogeneous net transfers, the state must supply strictly more goods $-P^*\delta_G$ than the equilibrium average net demand $N$ ($N + P^*\delta_g < 0$) in order to (strongly) determine the price level. Unlike in the situation with fiscal debtors in Proposition 6, the excess backing can be made arbitrarily small however, no matter the amount of heterogeneity across agents.

In the case of insufficient backing $N + P^*\delta_G \geq 0$, that we deem "financial repression", heterogeneity among net transfers creates gains from trades among creditors. Whereas there still exists an equilibrium without unofficial trades, there also exist equilibria with unofficial prices strictly above $N/(-\delta_G)$. In these equilibria, small creditors are willing to acquire money at a low price (at a high price level) in order to gain more dilution power in the official market. Conversely, agents with large claims accept to sell some money to them at such prices rather than further diluting their own positions in the official market. In other words, agents with low cash holdings arise as endogenous intermediaries between cash-rich agents and the state. We will show below that in the case in which $N + P^*\delta_G = 0$ , unofficial trades vanish in the limit of negligible agents.

**Symmetry with debt deflation.** Another way of describing such equilibria with multiple trading posts is that agents with little cash coordinate on cornering the official market by bidding borrowed cash. This forces agents holding more cash to sell it at a high unofficial price level, thereby financing the cornering strategy. These equilibria are thus symmetric to the debt-deflation ones in which agents with low fiscal debt corner the official market by flooding it with goods thereby forcing the more indebted ones to sell goods at a low unofficial price.

It is worthwhile noticing that in both cases, more unofficial trades yield more redistribution from the agents with the largest cash positions in absolute values towards the others, as the former force the latter to trade at unofficial prices that are less favorable than the official one.

**Capital controls as a complement to financial repression.** If financial repression is strict ($N + P^* \delta_G > 0$) and agents are ex ante identical, gains from trade between them vanish and so does the possibility of unofficial trades.[7] All agents would still be ready to sell some of their money holdings at a high price level. But nobody is willing to trade them for goods at this price as recycling money in the official market comes at the cost of diluting already large orders. This case of ex ante identical agents may be viewed as a strategic version of a representative-agent model. We show that in this case, financial repression works: The state can impose any price level no matter how little backing for its currency. Whereas we believe that this case of ex ante identical agents is of limited empirical relevance, it still generates the practical insight that financial repression is less likely to trigger the rise of unofficial prices if all the institutions that have access to the official market are long the local currency in similar magnitudes, as none of them can benefit from unofficial trade of a devalued currency. Capital controls can be viewed as an implementation of this situation.

**Asymptotically atomistic economies.** It is interesting to separate out, among the results in Proposition 7, the ones that survive in the limit in which each private agent becomes negligible. To be sure, the results that hinge on private agents' price impact are interesting in their own right, as there is ample evidence that the large institutions that participate in the primary markets for public liabilities have some price impact in practice. Yet assessing our results in the ideal case of negligible agents is also instructive. Here we show that as the economy converges to one in which each agent becomes negligible, the unofficial prices that may arise in the presence of financial repression all tend to $N/(-\delta_G) \geq P^*$.

**Proposition 8.** ***(Negligible agents)*** *Consider a sequence $(\mathcal{P}^n)_{n \in \mathbb{N}}$ of policies each associated with an economy of size $n$, and each such that the net transfers are not all*

[7] Even though it is not apparent given our binary definition of price-level determination, one can show that this result is actually continuous in nature: The more homogeneous the net transfers, the smaller the largest feasible unofficial trading volume.

*identical and* $0 < -P^{*,n}\delta_G^n \leq N^n$*. Suppose that* $(\mathcal{P}^n)_{n\in\mathbb{N}} \to \mathcal{P}$ *such that* $-P^*\delta_G > 0$ *and that*

$$\max_{i\in\{1,\ldots,n\}} \left\{ \frac{N_i^n}{nN^n} \right\} \underset{n\to+\infty}{\to} 0. \tag{24}$$

*For every* $\epsilon > 0$*, there exists* $m \in \mathbb{N}$ *such that for all* $n \geq m$*, the unofficial prices that are predictable given* $\mathcal{P}^n$ *belong to* $(N/(-\delta_G) - \epsilon, N/(-\delta_G) + \epsilon)$*.*

*Proof.* Conditions (14) and (24) together imply that every unofficial price must become arbitrarily close $P^*/\delta(P^*) = N/(-\delta_G)$ as $n \to +\infty$. □

Condition (24) encodes that agents become negligible in the limit. Notice that this proposition implies in particular than when the state backs money with the exact real amount given $P^*$—$N + P^*\delta_G = 0$—then all equilibrium prices converge to the official target $P^*$. If $N + P^*\delta_G > 0$, there still are multiple equilibria with varying trading volume at unofficial prices, including possibly no unofficial trade. Yet all unofficial prices become arbitrarily close to $N/(-\delta_G)$.

The coefficient $\Delta_i(P^*)$ defined in (12) that drives agents' indifference between distant prices illustrates the respective contributions of insufficient backing on one hand ($N + P^*\delta_G > 0$) and of the price impact of non-negligible agents on the other hand to the possible rise of unofficial trades. This coefficient is the product of the term $(d(P^*) - d_i(P^*))/d(P^*)$ that reflects individual price impacts and vanishes as agents become negligible, and of $s(P^*)/d(P^*)$. That $-\delta_G = s(P^*) < d(P^*) = N/P^*$ in the presence of financial repression is then the only remaining source of unofficial trade as agents become negligible.

**Remark on financial repression and the relevance of in-kind taxation.** In the absence of financial repression, condition (22) yields a standard (static) formula that does not directly depend on official trades, linking the real value of state liabilities and real surplus:

$$\sum_{i\in\mathcal{I}} L_i = P^*(f - nc_{G,C}), \tag{25}$$

where $f \equiv n\tau + (\sum_{i \in \mathcal{I}} T_i)/P^*$ are the real fiscal resources of the state. Thus, among the policies that determine the price level, those that differ only along the modalities of tax payment—in-kind versus in cash—but not along the real value of taxes nor along other dimensions lead to the same real allocations.

By contrast, the modalities of tax payment are no longer irrelevant under financial repression. As is transparent from expressions (22) and (23), a reduction in $\sum_{i \in \mathcal{I}} T_i$ and increase in $\tau$ holding $f$ fixed shifts real resources from the private sector towards the state since this reduces the private demand for money for tax-payment motives, and thus increases households' forced money holdings.[8] This is reminiscent of historical situations in which a financially distressed public authority imposed in-kind payments for some taxes (typically, on agricultural products), such as the Confederation during the US civil war or the USSR in the 20s.

**Revisiting the case $N_- N_+ > 0$.** It is easy to see that Propositions 6 and 7 together imply that if the state sells $-\bar{\delta}_G$ at $\bar{P}$ such that $N_+ + \bar{P}\bar{\delta}_G < 0$, and buys $\underline{\delta}_G$ at $\underline{P} < \bar{P}$ such that $N_i + \underline{P} \min\{\underline{\delta}_G, e - \tau\} \geq 0$ for all $i \in \mathcal{I}$, then the predictable prices must be within $[\underline{P}, \bar{P}]$, an interval that can be made arbitrarily small.

## 3.4 Feasible policies

We have thus far allowed for policies such that, depending on the strategy profile of the private sector, the state possibly ends up with strictly negative consumption of goods or/and money. Here we show that all the above results still hold for policies that we deem "feasible". These policies are such that the state's consumption is always positive no matter what the private sector does:

**Definition 6.** ***(Feasible policies)*** *A policy is feasible if and only if the state consumes positively goods and money for every private strategy profile.*

We have:

**Proposition 9.** ***(Characterization of feasible policies)*** *A policy is feasible if and*

[8] The state's consumptions given by (22) and (23) are constant across all equilibria with financial repression since unofficial markets do not affect them.

*only if*

$$\tau + \delta_G \geq 0, \tag{26}$$

$$M \geq N + P^* \min\left\{\delta_G^+, e - \tau\right\}. \tag{27}$$

*Proof.* **Positive consumption of goods.** The state transfers goods to the private sector only through sales, and not more than $-\delta_G$ per capita, ensuring that condition (26) is sufficient. Suppose that agent $i \in \mathcal{I}$ buys an arbitrarily small quantity at an arbitrarily large price from agent $j \in \mathcal{I}$ whom in turn bids the money, supposed to be larger than $-P^*\delta_G$, in the official market. Other agents do not trade. Agent $i$ also sells $e - \tau$ at an arbitrarily low price. Then the state must sell $\delta_G$ units and receives arbitrarily few goods from the possible bankruptcy of $i$, establishing that (26) is also necessary.

**Positive consumption of money.** The right-hand side of condition (27) corresponds to the amount of money that the state must transfer to the private sector when the latter sells as many goods as possible and pays its taxes. This is the maximum amount$M$ that the state needs to issue across all private profiles since the state issues additional money when agents default on their taxes by assumption. □

Conditions (26) and (27) merely state that the state has enough real resources $n\tau$ and prints enough money $nM$ to consume positively goods and money no matter the strategy profile of the private sector. Condition (26) ensures that the state consumes a positive quantity of goods no matter the private strategy profile. Condition (27) ensures that the state consumes a positive quantity of money for all private strategies.

It is important to stress that all the results up to Proposition 8 imply restrictions on all the components of a policy $\mathcal{P}$ except for $M$ and $\tau$. Thus all these results apply in particular to feasible policies, all that is needed is that $M$ and $\tau$ are taken sufficiently large that the feasibility conditions (26) and (27) hold for the policies of interest.

## 3.5 Defaultable security

The policy assumed thus far comprises a fixed nominal payment $L_i$ from the state to private agent $i \in \mathcal{I}$. This section considers policies in which the payments to the private sector are subject to default, in the sense that the aggregate payment cannot exceed the aggregate amount of money that the state collects by selling goods. In other words, the state does not use the money that it produces to honor its transfers but only the one that it purchases from the private sector.

We modify the baseline policy as follows. For simplicity, we assume away cash taxes: $T_i = 0$ for all $i \in \mathcal{I}$. More important, we suppose that the positive cash transfer from the state to agent $i \in \mathcal{I}$ depends on the amount of money that the state collects in the market. In game-theoretic language, this transfer is no longer an action of the state but rather a (negative) payoff per capita $L_i(\mathcal{S})$ that depends on the private strategy profile $\mathcal{S}$. Formally, there exists $(B_i)_{i\in\mathcal{I}}$, a positive vector such that $B = \sum_{i\in\mathcal{I}} B_i/n > 0$, and such that the transfer to agent $i \in \mathcal{I}$ is

$$L_i(\mathcal{S}) = B_i \min\left\{1, \frac{\Delta(\mathcal{S})}{nB}\right\}, \tag{28}$$

$$\textit{where } \Delta(\mathcal{S}) \equiv \left(\sum_{i\in\mathcal{I}} P^* \hat{d}_i(P^*) - P^* \hat{s}_i(P^*)\right)^+ \tag{29}$$

The variable $\Delta(\mathcal{S})$ is the nominal value, if positive, of the effective net demand of the private sector at the official post. Definition (28) implies that creditors are treated pari passu. This is to fix ideas, all our insights carry over with alternative seniority rules.

The rest of the policy is unchanged and so are the trading and bankruptcy mechanisms. For brevity we consider only the case $\delta_G < 0$. We have:

**Proposition 10.** ***(Defaultable security and price-level determination)*** *For all $D \in [(B + P^*\delta_G)^+, B]$, there exist equilibria in which the state pays $B - D$ per capita. Across such equilibria,*

$$c_{G,C} = \tau - \frac{B - D}{P^*}, \; c_{G,M} = M. \tag{30}$$

*There is no price-level determination. However, all active price levels converge to $P^*$ as agents become negligible in the sense of Proposition 8.*

*Proof.* Notice first that agents can always avoid bankruptcy by not trading. Notice also that there exists a no-trade equilibrium, so that determination of the price level is at best weak.

For every $D \in [(B + P^*\delta_G)^+, B]$, the situation whereby agent $i \in \mathcal{I}$ bids $B_i(1 - D/B)$ at $P^*$ and collects the same transfer from the state is clearly an equilibrium.

Suppose that an equilibrium associated with $D$ features another active post. Suppose that there is a net buyer at $P^b > P^*$. Lemma 3 applies with $P' = P^b$ and $P = P^*$. To see this, notice that the deviation used to prove the lemma—shifting part of the $P'$-bid towards $P$—strictly improves solvency as it increases the transfer received by the deviating agent. Thus condition (14) holds and $P^b$ becomes arbitrarily close to $P^*$ as agents become negligible. Suppose then that there is a net buyer below $P^*$, and let $P^b$ denote the smallest price at which there is one. There must be net sellers at this price. Lemma 2 applies between $P^b$ and any other active price $P \neq P^*$, implying that these net sellers, who must be buying somewhere else, buy at $P^*$. Suppose that such a net seller $i \in \mathcal{I}$ reduces her effective sales at $P^b$ by $\epsilon > 0$ sufficiently small. Her order at $P^*$ must then shrink by $x$ such that

$$P^* x = \epsilon P^b + \frac{B_i P^* x}{nB}, \tag{31}$$

where the second term on the right-hand side reflects that her smaller bid reduces her net transfer from the state, and thus tightens her solvency constraint. Equilibrium requires that $x \geq \epsilon$, or $P^b \geq P^*[1 - B_i/(nB)]$, and so, as $P^b < P^*$, $P^b$ must become arbitrarily close to $P^*$ as agents become negligible.

It remains to construct an equilibrium with unofficial trading. For $D$ such that $B - D + P^*\delta_g < 0$, let $i \in \mathcal{I}$ an agent with a minimum value of $B_i$. $i$ does not need to be unique. Let this agent buy a sufficiently small quantity at $P = P^*[1 - B_i/(nB)]$ and all the others sell their $e - \tau$ at this post and invest the proceeds at $P^*$. It is an equilibrium as the unofficial buyer is indifferent between buying at $P$ and $P^*$ and sellers may strictly prefer to sell more (unless their claim is equal to $B_i$ in which case they are indifferent) but cannot. □

Within the limits of a static model, the equilibria are reminiscent of self-fulfilling debt crises. Equilibria with default here are gridlocks whereby the private sector does not bid much cash for goods because it expects state default in the form of a small transfer, and these small bids in turn vindicate the small transfer. A key difference with the case of a nominally safe security is that there is no longer room for strict financial repression since the payment of the state by construction never exceeds the amount of money that it is willing to purchase at $P^*$. As a result, the situation bears similarities with that in which $N + P^*\delta_G = 0$ with safe securities: Unofficial trades are made possible only because of individual price impacts. Thus unofficial prices all become arbitrarily close to the official one when agents become negligible. Yet there are multiple equilibria even in this negligible limit, but only the goods consumption of the state varies across them. The price level does not. A higher loss given default creates additional real resources for the state.

It is interesting to notice that with defaultable securities, unlike with safe ones, we are able to construct unofficial trades at a lower price than the official target $P^*$ when away from the negligible limit. The reason agents are willing to sell at a lower price than $P^*$ and reinvest the proceeds at $P^*$ is that these proceeds benefit from a multiplier since by injecting more cash in the official market, payments alleviate their solvency constraint. This is because they collect back as a transfer a fraction of their bid. Thus the sales at unofficial prices earn a convenience yield associated with the fact that more demand in the official market makes all agents more solvent, including the seller. This effect vanishes in the negligible limit.

## 3.6 Market-clearing policy

The goal of this section is to compare the outcomes when the state's trading strategy consists in posting a fixed price as above with those when the state acts as an auctioneer à la Shapley and Shubik (1977), setting a price that absorbs all the private money supply in equilibrium. This will highlight the crucial role of the trading protocol on the set of predictable price levels.

We modify the baseline policy as follows. First, for brevity, we restrict again the analysis to policies such that $T_i = 0$ for all $i \in \{1, ..., n\}$ and $\delta_G < 0$.

Second, the official trading post no longer operates as the unofficial ones, but rather as a "sell-all" market à la Shapley and Shubik (1977). Private agent $i \in \mathcal{I}$ bids a positive quantity of money $C_i \geq 0$. The official price is then a function of the private strategy profile $\mathcal{S}$ defined as

$$P(\mathcal{S}) \equiv \frac{\sum_{i \in \mathcal{I}} C_i}{-n\delta_G}. \tag{32}$$

Finally, since our main goal is to highlight how defaultability and the trading protocol jointly determine the price level or fail to do so, we posit that the transfers to the private sector feature both a safe and a defaultable component. There exists a positive sequence $(l_i, B_i)_{i\in\mathcal{I}}$ such that $l = \sum_{i\in\mathcal{I}} l_i/n > 0$ and that the net transfer to agent $i \in \mathcal{I}$ is

$$L_i(\mathcal{S}) = l_i + \frac{B_i}{B} \min\left\{B, \left(\frac{1}{n}\sum_{i\in\mathcal{I}} C_i - l\right)^+\right\}, \tag{33}$$

where $B = \sum_{i\in\mathcal{I}} B_i/n$.

**Proposition 11.** ***(Market-clearing policy and price-level determination)*** *The set of predictable price levels includes* $[-l/\delta_G, -(l + B)/\delta_G]$. *For every* $D \in [0, B]$, *there exists an equilibrium with trades only in the official market at the price* $-(l+B-D)/\delta_G$, *and there exist equilibria in which such official trades coexist with active unofficial markets. Furthermore, across all equilibria* $c_{G,C} = \tau + \delta_G$, $c_{G,M} = M$.

*The set* $\Pi(\mathcal{P})\backslash[-l/\delta_G, -(l + B)/\delta_G]$ *becomes negligible as agents become negligible in the sense of Proposition 8.*

*Proof.* Notice first that there is always trade in equilibrium as $l > 0$ and $\delta_G < 0$. Second, for every $D \in [0, B]$, there exists an equilibrium whereby each agent $i \in \mathcal{I}$ bids $C_i = l_i+B_i(1-D/B)$ at the official post and the price is $-(l+B-D)/\delta_G$. One can also construct an equilibrium with a high unofficial price level similar to those when the official price is fixed. The reason is that an agent who buys dear has the same payoff from deviating towards the official post with a market-clearing price as with a fixed official price and uniform rationing. Formally, consider some agent $i \in \mathcal{I}$ and $0 < C_i < l_i + B_i(1 - D/B)$. Let us construct an equilibrium in which agent $i$ posts $C_i$ on the official market and sells a nominal value $l_i + B_i(1 - D/B) - C_i$ in an unofficial market with price $P$. All the other agents post all their goods on the unofficial market and are uniformly rationed. They also bid a nominal amount $N = \sum_{k\in\mathcal{I}} l_k + B_k(1 - D/B) - C_i$ in the official market. Let us denote $P(x) \equiv (N + C_i + x)/(-n\delta_G)$ the price on the official market when agent $i$ bids $x + C_i$ in the official market. The trading strategy for all agents except $i$ is optimal whenever $P > P(x = 0)$ and posting $C_i$ on the official market is optimal for agent $i$ when $0 = \arg\max(C_i + x)/P(x) + (l_i + B_i(1 + \min\{x; D\}/B - D/B) - C_i - x)/P$. The term $B_i/B \min\{x; D\}$ stems from the fact that bidding an additional $x$ on the official market leads to an additional $B_i/Bx$ units of money for agent i as debt repayment. When $D > 0$, when $N/(N + C_i) = (1 - B_i/B)P(x = 0)/P$, the global optimum is

$x = 0$. Otherwise, when $D = 0$, this condition is $N/(N + C_i) = P(x = 0)/P$. As these two conditions may hold jointly, this proves the existence of an unofficial market with exchange $l_i + B_i(1 - D/B) - C_i$ of money at the price $P = (N + C_i)^2/(-n\delta^G N)$.

Finally, the proof that unofficial prices converge to the official one when agents become negligible is similar to that in Proposition 10, and so we omit it. □

An implication of Proposition 11 is that a market-clearing policy does not prevent the emergence of unofficial markets, at least away from the limit of negligible agents. When some agents bid more money than what they initially have on the official market by buying money on unofficial markets, they can force at least one agent to sell part of her money at a higher price level. This higher price level makes this latter agent indifferent between this unofficial trade and a higher price impact were she all in in the official market.

**The change in trading protocol flips the relationship between default and price-level determination.** The salient implication of Proposition 11 is that shifting the official trading protocol from fixed price to market-clearing price flips the relationship between the defaultable nature of public liabilities and price-level determination in the negligible limit. Proposition 7 shows that the fixed-price trading protocol fails in general to determine the price level in the presence of a non-defaultable security because financial repression opens up the possibility of trade at multiple prices in equilibrium. Proposition 10 shows that by contrast, this fixed-price protocol determines the price level in the presence of a defaultable security in the negligible limit because only spending varies across equilibria with varying haircuts on public debt. The market-clearing trading protocol generates the exact opposite prediction. The price level is determined in the negligible limit if and only if the security is non-defaultable. In the presence of defaultable securities, it is the price level that absorbs all the fluctuations in the haircut on public debt across equilibria with varying default severity, whereas state spending remains unaffected.

In sum, in the negligible limit, safe securities warrant price-level determination in the presence of a market-clearing official price whereas defaultable ones do so when the official price is fixed.

### 3.7 Implications for fiscal and monetary interactions

This section discusses the implications of our results for fiscal and monetary interactions. It considers only our results under the limiting assumption of negligible private agents since the analysis of fiscal and monetary interactions is typically carried out in Walrasian environments. In this limit of negligible agents, our results regarding policies with fiscal creditors only can be summarized as follows. There are two situations in which there is determination of the price level in the negligible limit:

i) Money officially trades at a fixed price level and the state liabilities are either defaultable (Proposition 10), or safe but fully backed by goods (case $N + P^*\delta_G \leq 0$ in Proposition 7).

ii) Money officially trades at a market-clearing price and the liabilities are safe (case $B = 0$ in Proposition 11).

There are also two situations without determination:

iii) Money officially trades at a fixed price level and the state's liabilities are safe but there is financial repression (case $N + P^*\delta_G > 0$ in Proposition 7).[9]

iv) Money officially trades at a market-clearing price and the liabilities are defaultable (case $B > 0$ in Proposition 11).

These four situations admit interesting interpretations in terms of fiscal and monetary interactions. Viewing monetary policy as encompassing the decisions to issue and trade money, and fiscal policy as the determination of taxes and the issuance of other nominal liabilities, one can decompose a policy $\mathcal{P}$ into monetary policy—the issuance of $M$ and the trading of money given real resources $-\delta_G$, and fiscal policy—the determination of transfers and repayment of liabilities. Given this interpretation, one can identify two channels through which fiscal concerns may influence monetary policy in our setup. First, monetary policy may seek to ensure that the state does not default. Second, monetary policy may seek to make real spending $c_{G,C}$ insensitive to the (in and out of equilibrium) decisions of the private sector. One can then classify policies along the extent to which these two fiscal concerns influence monetary policy.

[9] Recall that there is a singular situation in which financial repression does not affect the determination of the price level because agents receive identical transfers.

With this respect, case i) with defaultable securities stands out as one of pure monetary dominance. The price level is determined because monetary policy consists only in trading money at its official price whether there is default or not and regardless of the impact of the private sector's decisions on spending. Case ii) is the other extreme in which the price level is determined because of fiscal dominance. Here the monetary authority prints and transfers enough money to make public liabilities safe, and sets the price level so that state spending is unaffected by private decisions.

Cases iii) and iv) are intermediate situations in which fiscal concerns influence monetary policy without fully dictating it. This leads to price-level indetermination. In case iii), the monetary authority prints enough money to avert sovereign default. Yet it sticks to its price-level target, thereby creating financial repression. The price level is indeterminate but all equilibrium price levels are a weighted sum of the official target $P^*$ and the unofficial price $N/(-\delta_G) > P^*$ that reflects insufficient backing. The weights vary across equilibria. In case iv), there is no money printing to avoid default, but the price level adjusts—price level and default intensity comove negatively—so that the state's consumption is not too dependent on the private sector's behavior and the severity of default.

# 4 Two-date model

This section analyzes a simple two-date extension of the one-date model. Its main goal is to study the extent to which future price-level determination or the lack thereof affects current price-level determination. Given this narrow focus, we consider a less general setup than the one discussed in the case of one date.

We now suppose that there are two dates $\{0, 1\}$. The $n \geq 2$ private agents value only a date-1 consumption good that is obtained out of the storage of a date-0 consumption good at a linear rate $\rho > 0$ between 0 and 1. Each agent $i \in \mathcal{I}$ is endowed with $e_i > 0$ units of the date-0 consumption good. We denote $e = 1/n \sum_{i \in \mathcal{I}} e_i$. We suppose that there exists $i \in \mathcal{I}$ such that $e_i \neq e$.

**Policy.** A policy $\mathcal{P} = (\delta_{G,0}, P_0^*, R, \delta_{G,1}, P_1^*)$ consists in two trades and one contingent transfer:

- **Trades.** The state stands ready to buy up to $n\delta_{G,0} > 0$ units of the date-0 good at a price $P_0^*$, and to sell up to $-n\delta_{G,1} > 0$ units of the date-1 good at a price $P_1^*$;

- **Transfers.** The state multiplies any outstanding net position in money by a private agent at the end of date 0 by $R > 0$, and this defines her net position at the outset of date 1.

For brevity, we study only policies that are not contingent at date 1 on the date-0 actions of the private sector.

**Private trades and bankruptcy.** At each date $t \in \{0, 1\}$, private agents can submit any number of buy or sell orders of the date-$t$ good, with the restriction that they cannot place sell orders for a total quantity larger than their endowment at the outset of each date. Trading posts clear with uniform rationing as in the one-date model.

With a straightforward extension of the one-date notations, the strategy of agent $i \in \mathcal{I}$ is $\mathcal{S}_i = (D_{0,i}(.), S_{0,i}(.), D_{1,i}(.), S_{1,i}(.))$. The date-1 orders are conditional on history, that is, on date-0 actions. Agent $i \in \mathcal{I}$ is bankrupt at date 0 if and only if

$$\int PdD_{0,i}(P) > \frac{1}{R}\left(\int Pd\hat{S}_{1,i}(P) - \int Pd\hat{D}_{1,i}(P)\right) + Pd\hat{S}_{0,i}(P), \tag{34}$$

and at date 1 if and only if

$$\int PdD_{1,i}(P) > R\left(\int Pd\hat{S}_{0,i}(P) - \int Pd\hat{D}_{0,i}(P)\right) + \int Pd\hat{S}_{1,i}(P). \tag{35}$$

**Equilibrium concept.** A profile $\mathcal{S} = (\mathcal{S}_i)_{i\in\mathcal{I}}$ is a predictable outcome given $\mathcal{P}$ if and only if it is a subgame-perfect Nash equilibrium.

The following proposition first characterizes the set of date-0 predictable price levels.

**Proposition 12.** ***(Date-0 outcomes)*** *Let*

$$r^* \equiv \frac{RP_0^*}{P_1^*}.$$

- *If $r^* = \rho$, the date-0 price level is determined, equal to $P_0^*$.*

- *If $r^* > \rho$, the date-0 price level is strongly determined and equal to $P_0^*$ if $e \leq \delta_{G,0}$. If $-\delta_{G,1}/r^* > e > \delta_{G,0}$, then it is not determined as there are equilibria with unofficial* *trade at prices below $P_0^*$.*

- *If $r^* < \rho$, the date-0 price level is not determined, there are equilibria with trades at unofficial prices. The set of predictable unofficial prices is $[P_0^*, \rho/r^* P_0^*)$.*

*Proof.* Suppose first that $r^* \geq \rho$. Notice that if $r^* > \rho$, then no trade is not an equilibrium: One agent could deviate and save at the official post. Let $\bar{P}_0$ the largest price at which there is active trading. We show that $\bar{P}_0 \leq P_0^*$. Suppose otherwise. Net buyers at $\bar{P}_0$ do not fund their purchases from selling at a lower date-0 price as this would be strictly suboptimal. Thus they must do so by selling at date 1 at a price $P_1 \geq R\bar{P}_0/\rho > r^* P_1^*/\rho \geq P_1^*$. Only net sellers at $\bar{P}_0$ are willing to be their counterparts at date 1 because other date-0 net sellers must buy at strictly lower prices to be solvent. But then the net buyers at $\bar{P}_0$ cannot recoup enough cash at date 1 because the net (date-0) sellers must place some buy orders strictly below $P_1$ at date 1 from Lemma 3, a contradiction.

If $\delta_{G,0} \geq e$ or $r^* = \rho$ there cannot be any active trade below $P_0^*$ because any seller there would be strictly better off with a deviation towards the official post. Finally, suppose $r^* > \rho$ and $\delta_{G,0} < e < -\delta_{G,1}/r^*$. We construct an equilibrium with unofficial trade at a price below $P_0^*$ as follows. For $\epsilon > 0$ sufficiently small, suppose that all agents sell their entire endowment at $P_0^*$ but one who sells all of it but an amount $\epsilon$ at $P_0$. The other agents bid the (effective) proceeds from their sales at this post. This is an equilibrium as soon as $P_0 = P_0^*[\delta_{G,0}/(e - n\epsilon) + \rho/r^*[1 - \delta_{G,0}/(e - n\epsilon)]]$.

Suppose then that $r^* < \rho$. Let $P_0 \in [P_0^*, \rho P_0^*/r^*)$. An agent $i \in \mathcal{I}$ buying some goods at $P_0$ from an agent $j \in \mathcal{I}$ at date 0, storing them and then selling them back to $j$ at $P_1 = RP_0/\rho = (r^*/\rho)(P_0/P_0^*)P_1^* < P_1^*$ is an equilibrium. There cannot be active trade at $P_0 < P_0^*$ because a seller on an unofficial post cannot earn more than $\rho$ and would thus be better off shifting part of her order at $P_0^*$ thus earning strictly more. There cannot be active trade at $P_0 \geq P_0^* \rho/r^*$, this stems from the same arguments as that establishing no unofficial trade above $P_0^*$ when $r^* \geq \rho$. □

When the state does not offer a sufficient return on money relative to storage ($r^* < \rho$), agents may opt for inside money. In this case, some agents buy at a higher date-0 price than the official one $P_0^*$ and store the purchased goods, financing the trade with a date-1 sale of the storage output at a lower price than $P_1^*$ at date 1. That $r^* < \rho$ implies that these agents can manufacture this way the same flows as that of a nominal debt contract with real return $\rho$, and their counterparts find this therefore equivalent to storage (and superior to money).

When money offers a weakly better return than storage, the price level is pegged at the official price when the state is ready to accept as many goods as the agents are willing to sell at this price. Otherwise, deflation may take place. This is a remarkable result implying that in this case, the date-0 price level is determined whether so is the date-1 price level or not. In fact, the following proposition shows that the price level may be determined this way at date 0 even when financial repression precludes determination at date 1.

**Proposition 13.** ***(Date-0 determination with or without date-1 determination)*** *Suppose $r^*[1 - e_i/(ne)] > \rho$ for all $i \in \mathcal{I}$ and $\delta_{G,0} \geq e$.*

**(i)** *The price level is strongly determined at date 0, equal to $P_0^*$.*

**(ii)** *If $-\delta_{G,1} > 0$ is sufficiently large or sufficiently small other things being equal, the date-1 price level is strongly determined, equal to $P_1^*$.*

**(iii)** *If $r^*e > -\delta_{G,1} > \rho/(1-1/n)\min_{i\in\mathcal{I}} e_i$, there is indetermination of the price level at date 1 with possible unofficial trades above $P_1^*$.*

*Proof.* The first point is a direct consequence from Proposition 12. Regarding the second point, if $-\delta_{G,1}$ is sufficiently large that $r^*e + \delta_{G,1} < 0$, all agents optimally invest their entire (heterogeneous) endowments at the official post at date 0 and their savings are fully backed at date 1, ensuring price-level determination from Proposition 7. If $-\delta_{G,1}$ is sufficiently small, all agents invest the same amount $x > 0$ such that $-\delta_{G,1}(1-1/n)/x = \rho$ at date 0. The condition for this to happen is that $x \leq \min_{i\in\mathcal{I}} e_i$. At date 1 they are in the situation in Proposition 7 in which they have identical savings and are strictly rationed, which implies price-level determination.

The last point covers the situations in which there is room for high unofficial prices at date 1. Let $y' = \min\{\max_{i\in\mathcal{I}} e_i, y\}$, where $y$ solves

$$-\delta_{G,1}\frac{\sum_{i\in\mathcal{I}}\min(e_i, y) - y}{(\sum_{i\in\mathcal{I}}\min(e_i, y))^2} = \rho, \tag{36}$$

or is equal to $+\infty$ if this equation admits no solution. The right-hand side of the condition defining case (iii) ensures that $y$ is larger than $\min_{i\in\mathcal{I}} e_i$. Agent $i \in \mathcal{I}$ invests $x_i =$

$\min\{e_i, y'\}$. The left-hand side of the condition defining case (iii) implies that agents are strictly rationed at date 1 with heterogeneous holdings and thus the case of financial repression in Proposition 7 applies. $\square$

Proposition 13 shows that the date-0 determination of the price level may or may not come with that of the date-1 price level. Even if the date-1 price level is not determined, by trading at a fixed price at date 0, the state ensures that the multiple date-1 equilibria translate into the indeterminacy of quantities, not prices at date 0.

The expectation of financial repression and the corresponding rationing at date 1 may still constrain portfolios. This happens in case (iii) when $-\delta_{G,1}[1 - e_i/(ne)]/e < \rho$ for some $i \in \mathcal{I}$. In this case, more affluent agents will optimally cap their holdings of money as rationing limits the return they obtain on money. To the extent that the cap does not prevent heterogenous portfolios, the resulting heterogeneity of money holdings leads to the possibility of unofficial trades as in Proposition 7. When $-\delta_{G,1}$ is small enough, as in case (ii), all agents cap their holdings of money leading to homogenous portfolios, which ensures price level determination.

# 5 Conclusion

This paper studies the extent to which distinctive capacities of the state—issuing money, declaring taxes, and implementing bankruptcy, together with its trades of money for other goods, imply that public financial policy determines the price level. Our concept of price-level determination is robust in the sense that we set all agents free to trade whichever quantities of money at whichever price level they want. In addition to characterizing policies that do determine the price level, we also offer a description of the set of predictable price levels even when this is not a singleton. We obtain realistic predictable outcomes such as the rise of unofficial prices and that of endogenous intermediaries in this case of price-level indetermination. Incidentally, we show that the Walrasian framework features built-in trading bans that impose an arbitrary form of price control both on the government and on the private sector. Its implications for the determination of the price level crucially rely on these arbitrary restrictions.

Our focus has been on economies in which neither money nor other public liabilities play an important role at overcoming frictions. A natural route for future research is to incorporate such a role in the analysis. This would in particular allow us to develop a normative analysis, assessing for example the welfare costs of price-level indetermination. Other situations that our strategically closed model is well-suited to study are that of the coexistence of multiple (private or/and public) monies. We also leave this for future work.

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