A State Theory of Price Levels*

Jean Barthélemy   Eric Mengus   Guillaume Plantin

January 8, 2025

Abstract

This paper studies whether public financial policy, defined as the collection of transfers and trades of money between public and private sectors, can determine the price level. In an economy in which all agents are free to set the prices at which they privately trade money for goods with each other, we identify policies that elicit a single equilibrium price level. For policies that fail to do so, for example because different official and unofficial prices may coexist in equilibrium, we still offer tight restrictions on the set of predictable price levels.

*Barthélemy: Banque de France, 31 rue Croix des Petits Champs, 75001 Paris, France. Email: jean.barthelemy@banque-france.fr. Mengus: HEC Paris and CEPR, 1 rue de la Libération, 78350 Jouy-en-Josas, France. Email: mengus@hec.fr. Plantin: Sciences Po and CEPR, 28 rue des Saints-Pères, 75007 Paris, France. Email: guillaume.plantin@sciencespo.fr. We thank participants in various seminars and in particular Michael Choi, Xavier Ragot, Guillaume Rocheteau, and Gilles Saint-Paul for very helpful discussions. Errors are ours. The views expressed in this paper do not necessarily reflect the opinion of the Banque de France or the Eurosystem.

1 Introduction

The determination of the prices of consumption goods in terms of money is a basic yet largely unsettled question in economics. It is for example telling that the mainstream monetary models that are currently used to inform policy do not determine the price level. Still, public financial policies implement trades and transfers of money with the private sector that presumably play an important role in this determination. First, public sectors use like other agents money as a medium of exchange for goods and services. Central banks in particular trade the money that they issue for other stores of value such as foreign currencies in foreign exchange interventions, or (public or private) fixed-income securities in open-market operations. Some of the prices at which such trades settle serve as important nominal anchors, and may thus be explicit targets for monetary policy. Historical and contemporaneous examples abound, including metallic standards, currency pegs, or the targeting of some short-term interest rates. Second, fiscal policy creates private tax liabilities that may also affect the determination of the price level, be it only because money is legal tender for these liabilities (e.g., Lerner, 1947; Smith, 1776; Starr, 1974). Finally, that public sectors can issue money in order to make good on their other (explicit and implicit) liabilities, such as sovereign debt, creates another possible connection between the value of money and the actions of the issuing state (e.g., Sargent and Wallace, 1981).

Does such a collection of trades and transfers of money suffice to determine the price level? This paper studies this old question in an economy that has a simple, yet we believe essential, feature: All agents are free to privately trade money for goods with each other at whichever prices they agree upon. Namely, subject to a resource constraint, each agent can submit buy or sell orders of any quantity of goods at any price. Orders at the same price clear with uniform rationing of the larger side. This unfettered free trade can be sophisticated in the sense that agents can act as leveraged intermediaries across such endogenously created trading posts, bidding in one post the expected proceeds from another one subject to a natural solvency constraint. A public financial policy determines the price level in this economy if all equilibrium trades occur at the same price.

In other words, we study whether the state can determine the price level using only (positive and negative) transfers and its own trades, without imposing any form of implicit or explicit trading ban nor price control on society. We think that this is the appropriate environment to study robust price-level determination. This generates new insights into the respective contributions of public financial policy to the determination of the price level, and that of other features of social interactions, such as restrictions on trading protocols.

Perhaps the best-known trading restriction that helps determining the price level is the cash-in-advance constraint. We abstract from it as agents in our economy can buy goods in a trading post with money that they simultaneously purchase in another trading post. A more pervasive restriction on trading protocols with less transparent implications is the assumption of a Walrasian environment. This imposes *de facto* strong restrictions on trade. It leaves agents, including the state, with no option but to submit demand schedules in a centralized official market. Enforcing such extreme restrictions to exchange seems out of reach in even the most controlled societies. We will see that these restrictions are important when one economic good is money, one agent is the state, and money is desirable to the other agents only insofar as the state trades it, and accepts it as legal tender for tax liabilities of its own making.

We carry out our analysis in an environment that is fully strategic, and we use Nash equilibrium as our concept of predictable outcome. Unlike with Walrasian environments, this enables a distinction between on one hand the policies that are feasible, and on the other hand the policies that determine the price level among these feasible ones. As we will show, some feasible policies that fail to determine the price level have interesting and plausible properties such as the coexistence of official and unofficial prices. In this case the price level is not determined but all the prices at which money may trade are fully characterized by public financial policy—the collection of trades and transfers between the state and the private sector.

We first study an elementary one-commodity one-date economy. In this economy, public financial policy has three central components: i) a maximum quantity of goods that the state is willing to trade for money, ii) a price at which the state is willing to trade goods for money, and iii) a vector of monetary transfers from the state to each private agent. We deem fiscal debtors the private agents who receive a negative transfer from the state, and fiscal creditors those who receive a positive transfer.

Fixed policies. We first study the natural situation in which both the price at which the state is willing to trade and the transfers are fixed, as opposed to contingent on the actions of the private sector. This is empirically relevant as a fixed official price resembles a currency peg, a metallic standard, or some of the allocation mechanisms currently used by central banks in their refinancing operations. Fixed transfers correspond to the repayment of nominally safe public liabilities issued in the (for now unmodelled) past such as central-bank reserves.

Our first key insight is that such a combination of fixed transfers and a fixed official trading price determines the price level if and only if it is impossible for any subgroup of private agents to place trades in the official market that would lead other agents to be rationed in it. A policy that does not satisfy this property opens up the possibility of situations in which some agents coordinate on squeezing the official market in order to induce the other agents to trade at unfavorable unofficial prices with them. In order to discourage such behavior, the state must commit to trade quantities of goods or money that are strictly larger than the ones it ends up trading in equilibrium.

More concretely, suppose for example that the large fiscal debtors in this economy cannot purchase enough money from the state to meet their tax liabilities if other agents coordinate on buying more money than they need in the official market. This might lead to a situation akin to debt deflation, whereby these distressed large fiscal debtors are willing to offload desirable commodities at a low unofficial price, and their counterparts use the cash that they obtain in the official market to snap up these cheap commodities. Symmetrically, in the presence of fiscal creditors, financial repression may create situations in which some agents sell goods at a high unofficial price level, and use the proceeds to increase their bids in the official market. The resulting crowding out of the other bidders justifies in turn their willingness to buy dear in the unofficial market. Examples of such parallel markets at inflated prices abound in practice, for example when a currency peg is no longer credible.

In sum, if public financial policy consists in a fixed official trading price and given fixed transfers, the determination of the price level requires that the quantities that the state stands ready to trade with the private sector be strictly larger than the ones that actually change hands in equilibrium.

Contingent policies. Then we study policies that are contingent on the actions of the private sectors in ways that are empirically relevant. Our contingent official price is a market-clearing price à la Shapley and Shubik (1977)—the price that makes the in and out of equilibrium private demand for goods equal to the maximum quantity that the government supplies. Our contingent transfers are defaultable payments, in the sense that the state never transfers more in aggregate than the amount of money that it collects from trading—that is, the state does not create money to honor its liabilities. We find that policies such that the price or/and the transfers are contingent this way never determine the price level. They always create room for trades at unofficial prices. However, if the official price clears the market whereas transfers are nominally safe, or if, conversely, the official price is fixed whereas transfers are defaultable, the unofficial prices all converge to a single price level when all private agents become negligible in the sense that the maximum price impact that each of them can have tends to zero.

By contrast, if both the official price and the transfers are contingent—that is, if policy features both a market-clearing price and defaultable transfers, then there is no determination of the price level, not even in this negligible limit. There are multiple equilibria akin to self-fulfilling debt crises whereby private agents demand very few goods in the official market because they expect low payments from the government. This is self-justified, and generates a continuum of equilibria across which price levels decrease with the severity of default.

Dynamics. We then write a two-date extension of our model in order to endogenize all transfers as resulting from voluntary private decisions to buy government claims. We obtain that the current price level may still be determined even if agents do not expect the price level to be determined in the future, for example because they expect financial repression. This is so when future equilibrium multiplicity translates into the current indeterminacy of the quantity of money traded rather than on its price.

Related literature. The title of this paper is an unsubtle reference to the state theory of money outlined in Knapp (1924). As epitomized by the opening sentence of the book—*"Money is a creature of law."*—the state theory of money contends that the state has a unique ability to impose something as money due to its legislative capacity. Our contribution is to formally study the extent to which this capacity may suffice to determine the price level. Here, the formalization of the distinctive capacity of the state is that it is the only agent which can print money, declare taxes, and expropriate bankrupt private agents.

Bassetto (2002) pioneers the strategic foundations of price-level determination by public financial policy. Its goal is to offer an example of an economy in which the fiscal theory of the price level applies. We share with him a strategically closed environment that highlights the importance of credible out-of-equilibrium actions in shaping equilibrium outcomes. By lifting his restrictions to centralized markets and cash-in-advance, we also generate a number of additional and, we believe, empirically relevant insights.

Our approach also has points of contact with the literature that endogenizes trading frictions as pure coordination failures in economies that are not plagued by exogenous informational or search frictions. Important contributions include Lagos (2000) and Burdett et al. (2001). One can view our results as identifying public financial policies that eliminate such coordination failures. We emphasize in particular the central role of the trading protocol selected by the government. In this sense, our approach applies to a context of endogenous frictions the approach pioneered by Hu et al. (2009) that endogenizes trading mechanisms in the presence of exogenous trading frictions.

The search literature has like us emphasized that the willingness of the state to back its money by accepting to trade it for desirable goods is important (Aiyagari and Wallace, 1997; Li and Wright, 1998). In our model without exogenous frictions such backing is simply a necessary condition for price-level determination. In these papers, this source of value for money coexists with its role of mitigating search frictions, and they show that more backing makes it easier to sustain the Pareto-dominant monetary equilibria.

Finally, given the central role of strategic exchange in our framework, we revisit the old and large literature on the strategic foundations of Walrasian equilibrium. A review is beyond the scope of this paper, important contributions include Dubey (1982) and the reference herein, Dubey and Shubik (1980), Schmeidler (1980), and Shapley and Shubik (1977). We show that when an economic good—money—is only desirable to optimizing agents as legal tender, the producer of this good still has the ability to coordinate private agents on a given nominal anchor.

The paper is organized as follows. Section 2 outlines a static model. Section 3 solves it. Section 4 discusses historical and current situations for which we believe our insights to be particularly relevant. Section 5 shows how our approach helps clarify—and offers rigorous foundations for—the ones based on the Walrasian equilibrium concept. Section 6 outlines and solves a two-date model. Section 7 concludes. Proofs follow the propositions because we find them most of them instructive, yet the paper is written so that they can be skipped in a first reading.

2 One-date model

This section outlines our simple one-date economy. It presents a baseline public financial policy that consists in fixed negative transfers (taxes), in fixed positive transfers that may be interpreted as extinguishments of nominal liabilities issued in an unmodelled past (reserves with the central bank or nominal bond), and in a commitment to trade a given maximum quantity of goods for money at a fixed official price. Section 3 will solve for the predictable price levels associated with such a baseline policy. It will also compare the outcome with that associated with policies that feature transfers or trading prices that are contingent on the actions of the private sector instead of being fixed as in this baseline one.

2.1 Setup

The economy comprises a public sector—"the state"—and $n \geq 2$ private agents indexed over $\mathcal{I} \equiv \{1, ..., n\}$. There are two divisible economic goods, one deemed "the good" and the other "money" henceforth. The good is intrinsically desirable to private agents whereas money is not. Each private agent thus ranks any bundles of the good and money using the standard ordering of their respective quantities of the good only.

The state can produce money at zero cost. Private agents cannot. Each private agent is endowed with $e > 0$ units of the good. All private agents and the state can trade money for the good according to a mechanism described below.

Public financial policy. The state enforces a policy that features monetary and in-kind transfers, money creation, and trade. We describe each component of a policy in turn.

Negative transfers (taxes). The state levies both in-kind and cash taxes:

- *In-kind taxes.* The state collects a tax of $\tau \in [0, e)$ units of the good on each individual.

- *Taxes paid in cash.* The state requires that each private agent $i \in \mathcal{I}$ pay a tax equal to $T_i \geq 0$ units of money.

To be sure, in-kind taxation is essentially absent in modern economies. One can however more broadly interpret $n\tau$ as the share of the economy's endowment owned by the state.[1]

Positive transfers. The state makes a cash transfer $L_i \geq 0$ to each agent $i \in \mathcal{I}$.

Money creation. Policy also features the production of $nM \geq 0$ units of money.

Trade. The state posts an order to buy a quantity $n\delta_G \in \mathbb{R}$ of the good at the price level $P^* > 0$, with the convention that this is a sell order if $\delta_G \leq 0$.

In sum, a policy consists in a vector $\mathcal{P} = (\tau, (T_i)_{i \in \mathcal{I}}, (L_i)_{i \in \mathcal{I}}, M, P^*, \delta_G)$.

The state also consumes $nc_{G,C} \in \mathbb{R}$ units of the good and $nc_{G,M} \in \mathbb{R}$ units of money. We do not model this consumption as a component of policy but rather as a payoff to the state determined by both policy and by the private sector's strategy profile as detailed below. Bassetto (2002), unlike us, models public spending as a decision that is not contingent on the private sector's strategy, but he posits that taxes, unlike here, are adjusting in response to (in and out of equilibrium) private strategies in order to maintain this fixed spending level. Both approaches are thus equivalent and merely reflect that since the state's surplus depends on voluntary trades by the private sector, then either taxes or expenditures (or both) must be modelled as contingent on actions by all agents—as payoffs rather than actions in a game-theoretic setting.

Net transfers. We will make intensive use of the following natural concepts of net transfers associated with a policy $\mathcal{P}$.

Definition 1. ***(Net transfers)*** *For all $i \in \mathcal{I}$, let $N_i = L_i - T_i$. Let*

$$N = \frac{1}{n} \sum_{i \in \mathcal{I}} N_i, \; N_+ = \frac{1}{n} \sum_{i \in \mathcal{I}} \max\{N_i, 0\}, \; \text{and} \; N_- = \frac{-1}{n} \sum_{i \in \mathcal{I}} \min\{N_i, 0\}. \tag{1}$$

[1] We entertain the interpretation of $n\tau$ as in-kind taxation because Section 4 shows that our setup can shed light on historical episodes of in-kind taxation by financially distressed states.

Notice that $N = N_+ - N_-$. In words, N is the net nominal transfer per capita, N_+ is the private net fiscal credit per capita (counting a net debt as zero), and N_- the absolute value of fiscal net debt per capita (counting a net credit as zero). In the following, we will deem "fiscal creditors" the agents such that $N_i > 0$ and "fiscal debtors" that for whom $N_i < 0$.

Private actions. Taking policy $\mathcal{P}$ as given, private agents play a simultaneous game whereby they make decisions to trade and pay taxes. We describe these decisions in turn, and the resulting payoffs.

Taxes. Each private agent $i \in \mathcal{I}$ decides on the amount of cash taxes $\hat{T}_i \geq 0$ that she pays to the state.[2]

Trades. Each agent can submit any number of orders to buy or sell a given quantity of goods at a given price. The only restriction is that the total size of her sell orders—the sum of the quantities of goods over all her sell orders—cannot exceed $e - \tau$. This is essentially a no short-sales constraint, as one cannot sell goods that one needs to buy. We will see below that money can by contrast be sold short.

The trading strategy of agent $i \in \mathcal{I}$ is conveniently described by the functions describing her cumulative orders. The respective cumulative buy and sell orders at prices (weakly) lower than P, $D_i(P)$ and $S_i(P)$ respectively, are increasing step functions over $[0, +\infty)$ satisfying:

$$D_i(0) = S_i(0) = 0, \tag{2}$$

$$\lim_{+\infty} S_i \leq e - \tau. \tag{3}$$

Let us denote for all $P > 0$, $d_i(P)$ and $s_i(P)$ the respective buy and sell orders of i at the price P:

$$d_i(P) \equiv \int \mathbb{1}_{\{p=P\}} dD_i(p), \;\; s_i(P) \equiv \int \mathbb{1}_{\{p=P\}} dS_i(p). \tag{4}$$

[2] Notice that unlike the cash taxes T_i, the in-kind taxes τ are automatically collected. This is consistent with the interpretation that $n\tau$ is just the state's real endowment, and will simplify the bankruptcy process.

In sum, the strategy of agent $i \in \mathcal{I}$ is $\mathcal{S}_i = (\hat{T}_i, D_i(.), S_i(.))$. Let $\mathcal{S} = (\mathcal{S}_i)_{i\in\mathcal{I}}$ denote the strategy profile of the private sector.

Market clearing and bankruptcy mechanism. We now describe how market clearing and a bankruptcy mechanism shape the payoff of each agent given a policy $\mathcal{P}$ and a strategy profile $\mathcal{S}$.

Market clearing. For all $P > 0$, let $d(P)$ and $s(P)$ denote the aggregate buy and sell orders at the trading post P:

$$d(P) = \sum_{i\in\mathcal{I}} d_i(P) + \mathbb{1}_{\{P=P^*\}}\delta_G^+, \;\; s(P) = \sum_{i\in\mathcal{I}}^{n} s_i(P) + \mathbb{1}_{\{P=P^*\}}(-\delta_G)^+ \tag{5}$$

If $d(P)s(P) = 0$, then no trade takes place. Otherwise, the smallest side of the market is fully executed and the other side is rationed pro rata the size of each order. Formally, each private agent $i \in \mathcal{I}$ buys and sells effective quantities $\hat{d}_i(P)$ and $\hat{s}_i(P)$ such that:

$$\hat{d}_i(P) \equiv d_i(P) \min\left\{1, \frac{s(P)}{d(P)}\right\}, \;\; \hat{s}_i(P) \equiv s_i(P) \min\left\{1, \frac{d(P)}{s(P)}\right\}, \tag{6}$$

and the same uniform rationing rule applies to the state at $P = P^*$. We respectively denote $\hat{D}_i(P)$ and $\hat{S}_i(P)$ the respective cumulative effective purchases and sales of agent $i \in \mathcal{I}$.

The following definition is natural and important. It states that a trading post is active if and only if at least one private agent strictly gains or loses goods in it.

Definition 2. ***(Active trading post, net buyer, net seller)*** *Agent $i \in \mathcal{I}$ is net buyer (respectively net seller) at the trading post P if and only if $\hat{d}_i(P) > \hat{s}_i(P)$ ($\hat{s}_i(P) > \hat{d}_i(P)$ respectively). The trading post P is active if and only if at least one agent is net buyer or net seller at P.*

An active trading post always features both at least one net buyer and one net seller by definition, but one of them can be the state.

Why uniform rationing? The only property of uniform rationing that is crucial for our results is that a larger bid generates other things being equal a larger allocation. Many other trading games share this property, including for example the sequential clearing of bids at the same price in random order.[3]

Bankruptcy mechanism. Suppose that agent $i \in \mathcal{I}$

- either places orders such that

$$\int PdD_i(P) > L_i - \hat{T}_i + \int Pd\hat{S}_i(P), \tag{7}$$

- or pays taxes $\hat{T}_i < T_i$,

then the state seizes all the goods and money of that agent and replaces her in the market. The state creates all the money that is needed to execute this agent's buy orders and/or to make up for $T_i - \hat{T}_i$.

In words, if an agent either places buy orders for a total amount larger than $L_i - \hat{T}_i$ plus the cash value of her *effective* sell orders, or/and defaults on her taxes, then she is bankrupt and does not consume anything.

In this Bertrand-Cournot market structure, unlike in Shapley-Shubik games, there is no cash-in-advance constraint: An agent can pledge in a post the cash proceeds from the simultaneous sales of her goods on another post. The bankruptcy rule means however that the agent must "mark to market" when doing so: She cannot make up arbitrary valuations of her marketed goods. Betting more cash than the effective proceeds from selling her goods is punished by bankruptcy.

This bankruptcy rule borrows from Dubey (1982). It plays two distinct roles. First, it ensures that there is no default contagion in the sense that an agent does not have to worry that her buying decisions may trigger a chain of defaults affecting her counterparties' orders. This is because the state steps in and makes good on the commitment of her counterparts if they fall short of cash because of her. Notice that if this was the sole purpose of the bankruptcy rule, it would be sufficient to posit that each agent can honor its effective orders and tax payments to avoid bankruptcy. Namely, one could substitute $\int Pd\hat{D}_i(P)$, weakly smaller, for $\int PdD_i(P)$ on the left-hand side of the bankruptcy condition (7).

[3]This example would require explicit assumptions on how private agents assess random consumption bundles.

Imposing that the whole buy orders of an agent, $\int PdD_i(P)$, rather than only the effective ones, $\int Pd\hat{D}_i(P)$, matter to trigger bankruptcy plays a second role. It creates a cost for each agent to dilute the others with arbitrarily large buy orders that will be only partially executed. To dilute other buyers this way, an agent must have enough collateral in the form of sufficient cash obtained either through transfers N_i (possibly negative) and effective sales. We would like to stress that it is the perfect-foresight nature of the model that leads us to impose this admittedly natural restriction on whole rather than only effective buy orders.[4]

An illustrative example. To see concretely how trading limits work, suppose that three agents A, B, and C trade. B sells one unit at a unit price and one unit at a price of 2. C buys 0.5 units at 2 and one unit at one. A sells one unit at 2. When the market at 2 clears, there is excess supply and A receives 0.5 units of cash from this market effectively selling only 0.25 units. A is bankrupt if she bids to buy more than $L_A - T_A + 0.5$ units in the price-1 market.

Payoffs. The payoffs associated with $\mathcal{P}$ and $\mathcal{S}$ result immediately from this trading structure and bankruptcy rule. An agent $i \in \mathcal{I}$ does not consume goods nor money if bankrupt, and consumes otherwise respective quantities of goods and money $c_{i,C}$ and $c_{i,M}$:

$$c_{i,C} = e - \tau + \int d\hat{D}_i(P) - \int d\hat{S}_i(P), \tag{8}$$

$$c_{i,M} = L_i - \hat{T}_i + \int Pd\hat{S}_i(P) - \int Pd\hat{D}_i(P). \tag{9}$$

Notice that these consumptions are positive by construction. The consumption of goods and money by the state in the absence of private bankruptcy, $c_{G,C}$ and $c_{G,M}$, are by

[4] It is possible to show that agents would spontaneously satisfy this constraint (7) with whole buy orders even if they faced only constraints with effective orders in an extension in which policy is uncertain and the state fully accommodates buy orders at P^* with some probability. Private agents would then dilute each other in this P^*-post using only cash that they effectively obtain in other posts in order to avoid bankruptcy risk if the state fully meets demand.

conservation of quantities:

$$nc_{G,C} = ne - \sum_{i\in\mathcal{I}} c_{i,C}, \tag{10}$$

$$nc_{G,M} = nM - \sum_{i\in\mathcal{I}} c_{i,M}. \tag{11}$$

Remark on positive state consumptions. These state consumptions are not necessarily positive. Section 3.4 introduces restrictions on policies such that the state consumes positively no matter the private strategy profile. We will deem policies that satisfy these restrictions "feasible". Our main results however hold for any policy, whether they are "feasible" in this sense or not, and so we will present them abstracting from these restrictions.

2.2 Some definitions

We model social interactions as a game between private agents given policy. It is thus natural to adopt Nash equilibrium as our concept of predictable outcome. Formally, given that private agents do not care about their consumption of money, an equilibrium associated with a policy $\mathcal{P}$ is a strategy profile $\mathcal{S}$ such that for every $i \in \mathcal{I}$, strategy $\mathcal{S}_i$ maximizes $c_{i,C}$ given other strategies $\mathcal{S}_{-i}$ and policy $\mathcal{P}$. This equilibrium concept yields a natural definition of predictable price levels:

Definition 3. ***(Predictable price levels)*** *A price $P > 0$ is predictable given policy $\mathcal{P}$ if and only if there exists an equilibrium associated with $\mathcal{P}$ with active trading at P. Let $\Pi(\mathcal{P})$ denote the set of predictable price levels associated with a policy $\mathcal{P}$.*

This enables us in turn to characterize whether a public financial policy determines the price level:

Definition 4. ***(Determination of the price level)*** *A policy $\mathcal{P}$ weakly determines the price level if and only if $\Pi(\mathcal{P})$ is a singleton. A policy strongly determines the price level if and only if it weakly determines the price level and every equilibrium features active trade.*

The price level may fail to be determined for three reasons. First, it may be that there exists no equilibrium with active trade. Second, it may be that every equilibrium features active trade at a given equilibrium price, but that this latter price varies across equilibria. Finally, an equilibrium may feature active trades at different prices. We will see that there exist policies leading to each of these three configurations, together with the ones that actually determine the price level.

3 Analysis

We solve for the predictable price levels $\Pi(\mathcal{P})$ associated with a policy $\mathcal{P}$. The following section first introduces important properties of equilibrium trades.

3.1 Some properties of equilibrium trades

The following lemma first shows that one can offset trades by the same agent at a given price in the following sense.

Lemma 1. ***(Netting)*** *Consider a strategy profile such that agent $i \in \mathcal{I}$ is a non-bankrupt net buyer at the trading post P. If she deviates and sets $s'_i(P) = 0$, $d'_i(P) = d_i(P) - d(P)s_i(P)/s(P)$ then she does not affect her allocation nor that of other agents. Symmetrically, suppose she is net seller at P. If she deviates and sets $d'_i(P) = 0$, $s'_i(P) = s_i(P) - s(P)d_i(P)/d(P)$ then she does not affect her allocation nor that of other agents.*

Proof. The results stem directly from the fact that these deviations do not affect the rationing coefficients as when $s(P) \neq s_i(P)$ and $d(P) \neq d_i(P)$,

$$\frac{d(P) - \frac{d(P)si(P)}{s(P)}}{s(P) - s_i(P)} = \frac{d(P) - d_i(P)}{s(P) - \frac{s(P)di(P)}{d(P)}} = \frac{d(P)}{s(P)}.$$

The effective trades of the other agents are thus unaffected by the deviation of i. Nor is that of i. To see this, suppose that i is net buyer. The reduction in her buy order $d(P)s_i(P)/s(P)$ is weakly larger than that of her effective sales $s_i(P)\min\{d(P)/s(P), 1\}$, and so the deviation leaves her solvent. Furthermore,

$$\hat{d}'_i(P) - \hat{s}'_i(P) = \left(d_i(P) - d(P)\frac{s_i(P)}{s(P)}\right)\min\left\{1, \frac{s(P)}{d(P)}\right\} = \hat{d}_i(P) - \hat{s}_i(P),$$

leaving her allocation unchanged. The same reasoning applies for a net seller. □

This result is useful because it implies that whenever an agent is net seller or net buyer at one post, we can assume that she nets her trades this way before entering into a deviation so that we do not have to worry about the impact of small deviations from her larger effective order on her potential order on the other side.

For any active trading post P and any $i \in \mathcal{I}$, let us now define:

$$\Delta_i(P) = \begin{cases} \frac{s(P)(d(P)-d_i(P))}{d(P)^2} \text{ if } s(P) \leq d(P), \\ 1 \text{ otherwise.} \end{cases} \tag{12}$$

The coefficient $\Delta_i(P)$ measures the marginal return from increasing a buy order in a market in which buyers are (weakly) rationed ($s(P) \leq d(P)$). On one hand a marginal increase ϵ in i's order generates $\epsilon s(P)/d(P)$ additional marginal units. On the other hand it crowds out her outstanding order $d_i(P)$, thereby costing a marginal reduction $\epsilon(d_i(P)/d(P)) \times (s(P)/d(P))$ in the return on this outstanding order.

The following two lemmas introduce some important characteristics of equilibrium trades at multiple prices.

Lemma 2. ***(Selling high to buy low)*** *Suppose that in an equilibrium that features (at least) two active trading posts with price levels P and P', a non-bankrupt agent $i \in \mathcal{I}$ is net seller at P' and net buyer at P. Then $P' > P$, and if $s(P) < d(P)$,*

$$P'\Delta_i(P) \geq P. \tag{13}$$

Proof. We show that if $P' \leq P$ or $s(P) < d(P)$ and condition (13) does not hold, i can strictly increase her utility by simultaneously reducing her sell and buy positions. Let us define:

$$\delta(P, \epsilon) = \min\left\{1, \frac{s(P)}{d(P)+\epsilon}\right\} \text{ and } \sigma(P, \epsilon) = \min\left\{1, \frac{d(P)}{s(P)+\epsilon}\right\}.$$

We let i modify her orders as follows. She first nets her positions as in Lemma 1. If she is the only seller at P' and is rationed, she also reduces her order up to the total buy order. For notational simplicity we maintain the notation $s_i(P')$, $d_i(P)$ for these new orders.

For $\epsilon > 0$ sufficiently small, define $\eta(\epsilon)$ as

$$-P\eta(\epsilon) = P'[(s_i(P') - \epsilon)\sigma(P', -\epsilon) - s_i(P')\sigma(P', 0)]. \tag{14}$$

In words $\eta(\epsilon)$ is the reduction in the buy order at P that has the same monetary value $P\eta(\epsilon)$ as that in the reduction in effective sales at P' when the sell order is reduced by ϵ. In particular, $\eta(\epsilon) = \epsilon P'/P$ when $\sigma(P', 0) = 1$. Suppose that i reduces her sell order at P' by ϵ and her buy order at P by $\eta(\epsilon)$. This leave her solvent by construction of $\eta(\epsilon)$, and brings a net change in consumption:

$$\begin{aligned} &(d_i(P) - \eta(\epsilon))\delta\left(P, -\eta(\epsilon)\right) - d_i(P)\delta\left(P, 0\right) - (s_i(P') - \epsilon)\sigma(P', -\epsilon) + s_i(P')\sigma(P', 0) \\ =&d_i(P)(\delta\left(P, -\eta(\epsilon)\right) - \delta(P, 0)) + \eta(\epsilon)\left(\frac{P}{P'} - \delta\left(P, -\eta(\epsilon)\right)\right). \end{aligned}$$

At first-order in $\eta(\epsilon)$, this is equal to

$$\eta(\epsilon)\left(\frac{P}{P'} - \delta(P) + \mathbb{1}_{\{\delta(P)<1\}}\delta(P)\frac{d_i(P)}{d(P)}\right),$$

Thus this deviation yields a strict benefit if $P' < P$ or if $\delta(P) < 1$ and (13) does not hold, a contradiction. □

Intuitively, selling high to buy low is profitable only if the marginal redeployment of the sales proceeds to buy in the cheap trading post does not crowd out the outstanding order at this post. The following lemma offers a necessary condition for a private agent being willing to buy at the highest of two prices.

Lemma 3. ***(Buying high instead of low)*** *Suppose that an equilibrium features (at least) two active trading posts with price levels P and $P' > P$. If a non-bankrupt agent $i \in \mathcal{I}$ is net buyer at P' then*

$$P'\Delta_i(P) \leq P\Delta_i(P'). \tag{15}$$

Proof. We show that if condition (15) does not hold, i can strictly increase her utility by moving some of her P'-order at P. Notice that Lemma 2 ensures that i cannot be a net seller at P as $P' > P$. We let i modify her orders as follows. She first nets her positions as in Lemma 1. Then she reduces her buy order at P' by ϵ and increases her buy order at P (possibly equal to 0) by $\epsilon P'/P$, where $\epsilon > 0$ is sufficiently small. Her solvency constraint still holds since the total cash value of her buy orders is unchanged. The net change in consumption units resulting from this deviation is

$$\begin{aligned}
&(d_i(P') - \epsilon)\delta(P', -\epsilon) - d_i(P')\delta(P') + \left(d_i(P) + \epsilon\frac{P'}{P}\right)\delta\left(P, \epsilon\frac{P'}{P}\right) - d_i(P)\delta(P) \\
=&\epsilon\left(\frac{P'}{P}\delta\left(P, \epsilon\frac{P'}{P}\right) - \delta(P', -\epsilon)\right) + d_i(P)\left(\delta\left(P, \epsilon\frac{P'}{P}\right) - \delta(P)\right) \\
&+ d_i(P')\left(\delta\left(P', -\epsilon\right) - \delta(P')\right).
\end{aligned}$$

At first-order w.r.t. ϵ this is equal to

$$\epsilon\frac{P'\delta(P)}{P}\left[1 - \mathbb{1}_{\{\Delta_i(P)<1\}}\frac{d_i(P)}{d(P)}\right] - \epsilon\delta(P')\left[1 - \mathbb{1}_{\{\Delta_i(P')<1\}}\frac{d_i(P')}{d(P')}\right],$$

strictly positive if condition (15) does not hold, which establishes the result. □

Intuitively, an agent is willing to be net buyer at $P' > P$ if her order at P is sufficiently large that she would crowd herself out by rebalancing some of her expensive order P' towards P.

3.2 Some necessary conditions for price-level determination

Next, we identify two properties of policies that each entails that the policy fails to determine the price level. First, and not surprisingly, the absence of net transfers implies indetermination of the price level because there exists no equilibrium with active trading in this case.

Lemma 4. ***(No price-level determination without transfers)*** *If a policy is such that $N_+ + N_- = 0$ there is no determination of the price level because there is no equilibrium with active trading: $\Pi(\mathcal{P}) = \emptyset$.*

Proof. Suppose by contradiction that agent $i \in \mathcal{I}$ is active in at least one trading post. She cannot be net buyer in every post in which she is active since she would then be bankrupt from $\int PdD_i(P) \geq \int Pd\hat{D}_i(P) > \int Pd\hat{S}_i(P) + N_i$, and thus better off not trading at all. This implies that if there exists at least one active trading post, at least one private agent is net seller somewehere. Let $\underline{P}$ denote the smallest price at which there is a private net seller. She must be net buyer somewhere else otherwise she would be strictly better off not trading. From Lemma 2, it has to be at a lower price, but since $\underline{P}$ is the smallest price at which there is a private net seller, the only possible net seller facing her is the government, and she buys at $P^* < \underline{P}$. But then this means there is a private net buyer at $\underline{P}$, as it cannot be the state which buys at this price. Let $\bar{P} \geq \underline{P}$ denote the largest price at which there is a private net buyer. A net buyer at this price cannot be net seller at any lower price from Lemma 2. But then she must be bankrupt, a contradiction. □

Second, and more interestingly, we show that any policy that creates gains from trade between private agents because some are fiscal debtors and other fiscal creditors fails to determine the price level.

Lemma 5. ***(Private gains from trade preclude the determination of the price level)*** *If a policy is such that $N_+N_- > 0$ then it does not determine the price level because $\{P^*\} \subsetneq \Pi(\mathcal{P})$.*

Proof. Without loss of generality, we suppose that $(N_i)_{i\in\mathcal{I}}$ is increasing. We denote n_- the fiscal debtor with the smallest debt (the smallest absolute value of $N_i < 0$). Notice first that there always exists an equilibrium with a single active trading post at P^* in which agent $i \in \mathcal{I}$ submits a buy order N_i/P^* if $i > n_-$ and sells $\min\{e - \tau, -N_i/P^*\}$ otherwise. This "P^*-equilibrium" features no bankruptcy if and only if $P^*(e-\tau)+N_i \geq 0$ for all $i \in \{1, ..., n_-\}$ and $N + P^*\delta_G \geq 0$. The first condition states that every fiscal debtor has enough goods to sell to acquire $-N_i$ of money. The second one ensures that the demand of money by fiscal debtors is covered by the creditors' and public supply at P^*.

We construct another equilibrium in which there is active trade at two prices, P^* and $P > P^*$. We construct the equilibrium supposing that $N_n > N_{n-1}$. We explain how to adapt the analysis to the case in which several agents share this same highest value of net transfers N_n in Step 3 below.

Step 1. Suppose first that the P^*-equilibrium features no bankruptcy. We construct an equilibrium in which agent n invests a sufficiently small (in a sense made precise below) nominal amount B in a trading post $P > P^*$. All the other agents are on the other side of the market at P. The other fiscal creditors (if any) redeploy in the P^*-post the proceeds from selling their entire net endowment $e - \tau$ at P. The fiscal debtors mix sales at P^* and at the rationed higher price P so as to meet their liabilities at the lowest cost.

We first define fiscal debtors's strategies. For $B > 0$ sufficiently small, define $S(B)$ the positive solution to

$$\frac{n_-(e-\tau) - S(B)}{(n-1)(e-\tau) - S(B)} B + P^* S(B) = nN_-. \tag{16}$$

For B sufficiently small, for every $i \in \{1, ..., n_-\}$, there exists a strictly positive solution $s_i(P^*)$ to

$$\frac{(e-\tau) - s_i(P^*)}{(n-1)(e-\tau) - S(B)} B + P^* s_i(P^*) = -N_i, \tag{17}$$

and by definition

$$\sum_{i=1}^{n_-} s_i(P^*) = S(B). \tag{18}$$

The equilibrium is then such that fiscal debtor $i \in \{1, ..., n_-\}$ sells $s_i(P^*)$ at the P^*-post and $e - \tau - s_i(P^*)$ at the P-post where P is defined below.

Let us now define fiscal creditors' strategies. Agent $j \in \{n_- + 1, ..., n-1\}$ (if any) sells $e - \tau$ at P and invests a nominal amount equal to the proceeds plus N_j at P^*. Agent n invests a nominal amount $N_n - B$ at P^* and B at P. The supply at P^* is thus $s(P^*) = n(-\delta_G)^+ + S(B)$, the demand $d(P^*) = n\delta_G^+ + nN_+/P^* - B(n_-(e-\tau) - S(B))/[P^*[(n-1)(e-\tau) - S(B)]] = n\delta_G^+ + nN_+/P^* - nN_-/P^* + S(B)) \geq s(P^*)$. Let us define

$$P = \frac{P^* d(P^*)^2}{s(P^*)\left(d(P^*) - \frac{N_n - B}{P^*}\right)}. \tag{19}$$

Suppose B is sufficiently small that $N_n - B > N_{n-1} + B(e-\tau)/[(n-1)(e-\tau) - S(B)]$. Then n's trade is optimal from (19) and Lemma 3. So are the trades of the other fiscal creditors because Lemma 2 and (19) imply that they would like to sell more at P to reinvest at P^* but they hit their maximum supply $e - \tau$ at P. Finally, fiscal debtors cannot meet their net liabilities at a lower cost as they sell as much as possible at $P > P^*$ subject to being solvent.

Step 2. Suppose now that the P^*-equilibrium features at least one bankrupt agent because there exists $i \in \{1, ..., n_-\}$ such that $P^*(e - \tau) < -N_i$ or because $N + P^*\delta_G < 0$. We re-create essentially the same equilibrium as in Step 1. First, for any fiscal debtor $i \leq n_-$ such that $P^*(e - \tau) < -N_i$, replace $-N_i$ with $P^*(e - \tau)$. Second, take one bankrupt agent, and make him add a buy order larger than N_n/P^* (which of course will be executed by the state) at P^* such that overall $N' + P^*\delta_G \leq 0$ where the new aggregate transfer per capita N' factors in the revised sell and buy orders of the bankrupt agents. It is easy to see that replacing this buy order of the bankrupt agent by another one split between P^* and P defined as in Step 1 for B sufficiently small is an equilibrium.

Step 3. In order to adapt the proof to the case in which $k > 1$ agents share the same maximum transfer N_n, we leave it to the reader to check that one only needs to let each of them invest a nominal amount B/k in a P-post defined as in Step 1.

Remark on other equilibria. Given that the goal of the proof is to offer one example of indetermination, we focused on a particular equilibrium with multiple active prices that has the advantage of being sustainable for any policy such that $N_-N_+ > 0$. It is also a particular equilibrium that we will repeatedly use in the balance of the paper. To be sure, there are in general plethora of other equilibria, including some with active trade at lower prices than P^*. Characterizing them further is not in the scope of this paper. As an illustration of this multiplicity, it is easy to see that in the case in which $\delta_G = 0$, there exists an equilibrium with a single active post with price P for any $P > 0$. □

Lemma 5 states that if a policy opens up potential gains from trade between private agents because the transfers create both fiscal creditors and fiscal debtors, then it cannot determine the price level. Accordingly, in the remainder of the analysis, we will first focus on policies such that all net transfers have the same sign ($N_+N_- = 0$). We will then show that the insights that we obtain in such situations extend naturally to policies that create both fiscal creditors and debtors, provided one extends our baseline policies to ones with two official trading posts in opposite directions.

The essential reason private gains from trade make it impossible to peg the value of money with a single trade is that money serves no other purpose than dodging bankruptcy in this economy.[5] Thus fiscal creditors are happy to trade money for goods at any price. Symmetrically, debtors are happy to trade goods for money at any price provided this makes them solvent. (They also are indifferent between any trade in the absence of any way out of bankruptcy.) The single trade of the state is thus not sufficient to coordinate the private sector on its price-level target P^*. In the presence of gains from trade between them, private agents can always simultaneously trade on this official market and on unofficial ones at different price levels. Again, we will see in Section 3.3 that in this case in which $N_-N_+ > 0$, the state can essentially determine the price level with two trades in opposite directions.

3.3 Characterization of price-level determination

Having eliminated feasible policies that fail to determine the price level, we now characterize the ones that do. Lemmas 4 and 5 imply that a policy that determines the price level must be such that $N_+ + N_- > 0$ and $N_+N_- = 0$. In words, there must be net transfers and they must all have the same sign. Consider first the case in which there are only fiscal debtors. In this case, the following proposition shows that the out-of-equilibrium behavior of the state may have to deviate very significantly from the equilibrium one in order to ensure price-level determination.

Proposition 6. ***(Fiscal debtors and debt deflation)*** *Suppose that a policy $\mathcal{P}$ is such that $N_- > 0$ and $N_+ = 0$. There exist equilibria without bankruptcy if and only if*

$$N_i + P^*(e - \tau) \geq 0 \text{ for all } i \in \mathcal{I} \text{ and } N + P^*\delta_G \geq 0. \tag{20}$$

In any equilibrium without bankruptcy, active prices are in $(0, P^*]$.

Condition (20) does not suffice to ensure price-level determination. A sufficient condition for strong price-level determination with $\Pi(\mathcal{P}) = \{P^*\}$ is

$$N_i + P^* \min\{\delta_G, e - \tau\} \geq 0 \text{ for all } i \in \mathcal{I}. \tag{21}$$

Proof. We proceed in four steps.

[5] Section 6 develops a two-date version of the model in which money may also be desirable as a store of value.

Step 1: In any equilibrium without bankruptcy, active prices are in $(0, P^*]$. Suppose that an equilibrium is without bankruptcy. This implies that there must be active trading. Suppose that the highest active-trading price is strictly above P^*. Any net buyer at this price is a private agent and is not net seller anywhere from Lemma 2. But then she must be bankrupt, a contradiction.

Step 2: There exist equilibria without bankruptcy if and only if (20) holds. There always exists an equilibrium with a single trading post at P^* in which agent $i \in \mathcal{I}$ sells $\min\{e-\tau; -N_i/P^*\}$. This "P^*-equilibrium" features no bankruptcy if condition (20) holds because every agent can afford her taxes in this case. If (20) does not hold, any equilibrium without bankruptcy would require that the agents that are bankrupt in the P^*-equilibrium can sell goods at a strictly higher price than P^*, a contradiction from the above point.

Step 3: Condition (20) does not suffice to ensure price-level determination. We build a simple counter-example. Suppose $n > 2$, and, without loss of generality, that $(N_i)_{i \in \mathcal{I}}$ is increasing. Suppose that condition (20) holds, so that the P^*-equilibrium involves no bankruptcy. Suppose however that there exists $m \in [1, n-2]$ such that $(n-m+1)N_m + nP^*\delta_G < 0$. Notice that the existence of such an m given condition (20) implies that the fiscal debts of agents $i > m$ be sufficiently small in absolute values and $N + P^*\delta_G$ be sufficiently close to 0. If $e - \tau$ is sufficiently large other things being equal, there also exists an equilibrium in which all agents $i \leq m$—the "large" fiscal debtors—are bankrupt and sell their entire endowments at an arbitrarily small unofficial price to agents $j > m$—the "small" fiscal debtors. These latter small debtors bid their entire endowment at the official post and reinvest the proceeds at this unofficial low price. To see why this is an equilibrium, notice first that for $e - \tau$ sufficiently large, the small fiscal debtors squeeze the official market in this equilibrium. Thus a large fiscal debtor i would have a strict gain from deviating and buying cash on the official market to get out of bankruptcy if $nP^*\delta_G/(n-m+1) \geq -N_i$, which does not hold. Small fiscal debtors strictly benefit from this trade for $e - \tau$ sufficiently large as they give up $\delta_G n/(n-m)$ consumption units in the official market and get $(e-\tau)m/(n-m)$ in the unofficial one. Any of them would thus be strictly worse off just buying money in the official market to pay taxes and consuming strictly less than $e - \tau$.

Step 4: There is strong price-level determination if (21) **holds.** We show that the P^*-equilibrium is the only equilibrium if condition (21) holds. In this case, notice first that there is no equilibrium without active trading otherwise any agent such that $N_i < 0$ would be better off deviating and escaping bankruptcy by selling $-N_i/P^*$ at P^*. In any equilibrium in which there is trade at another price than P^*, there has to be a private net buyer and a private net seller. Let $\underline{P}$ denote the lowest price at which there is a private net seller i and $\bar{P}$ denote the highest price at which there is a private net buyer j. Net buyer j cannot sell at any lower price than $\bar{P}$ from Lemma 2 but must sell somewhere to avoid bankruptcy, which she could always achieve from condition (21). Thus she must sell at $P^* > \bar{P}$, which implies $\underline{P} \leq \bar{P} < P^*$. But then i, who is not net buyer at any post from condition (13) and $P^* > \underline{P}$, would be strictly better off selling only at P^* the amount required to pay her taxes, a contradiction. Condition (21) warrants that she is never too diluted by the other orders to achieve this. □

To grasp the intuition for the results behind Proposition 6, it is useful to start with the remark that there always exists an equilibrium with a single trading post at P^*, in which agent $i \in \mathcal{I}$ sells $\min\{e - \tau; -N_i/P^*\}$. We deem this equilibrium the "P^*-equilibrium".

If condition (20) holds, this equilibrium is without bankruptcy since i) each private agent has enough goods to sell to pay her net taxes ($-N_i > P^*(e-\tau)$), and ii) and their aggregate demand for money $-N$ is within the state's maximum supply $P^*\delta_G$. Proposition 6 shows that condition (20) is actually necessary for the existence of a bankruptcy-free equilibrium.

Debt-deflation equilibria. Interestingly, in terms of price-level determination, whereas condition (20) warrants that all predictable prices are smaller than the official target P^*, it does not suffice to rule out lower unofficial prices. As showcased by the example constructed in the proof, if (20) holds but i) fiscal debts are sufficiently heterogeneous and ii) the maximum supply of money $P^*\delta_G$ is sufficiently close to the P^*-equilibrium demand N, then equilibria that we deem ones of "debt deflation" may arise. In these equilibria, the agents with low fiscal debt coordinate on squeezing the official market, purchasing more money than they need so as to ration the larger fiscal debtors. The former can then redeploy this cash in an unofficial market in which they snap up goods sold by the latter distressed large fiscal debtors at a low price.

"Whatever it takes." The state can eliminate such debt-deflation equilibria by merely increasing δ_G, or committing to buy more goods thereby injecting enough money in the economy so that condition (21) holds. If for some reason it is unwilling to do so, it must hope for the private sector to coordinate on the P^*-equilibrium instead of one with debt deflation. The interpretation of condition (21) ensuring price-level determination is that the state commits to do whatever it takes to ensure that each single fiscal debtor can purchase money to honor her liabilities regardless of the (in or out of equilibrium) actions of the rest of the private sector. This implies standing ready to sell more money than the equilibrium quantity N.

Consider now the situation in which there are only fiscal creditors.

Proposition 7. ***(Fiscal creditors and financial repression)*** *Suppose that a policy $\mathcal{P}$ is such that $N_+ = N > 0$ and $N_- = 0$. There are three types of predictable outcomes:*

1. ***No active trade.*** *If $\delta_G \geq 0$, then $\Pi(\mathcal{P}) = \emptyset$.*

2. ***Strong price-level determination.*** *If $N < -P^*\delta_G$, then $\Pi(\mathcal{P}) = \{P^*\}$ and the policy strongly determines the price level. Furthermore,*

$$c_{G,C} = \tau - \frac{N}{P^*},\ c_{G,M} = M. \tag{22}$$

3. ***Financial repression.*** *If $N \geq -P^*\delta_G > 0$, then there is strong determination of the price level iff $N_i = N$ for all $i \in \mathcal{I}$. Otherwise, there also exist equilibria with multiple active trading posts, with all unofficial prices strictly above $N/(-\delta_G)$. Whether the price level is determined or not, it is always the case that:*

$$c_{G,C} = \tau + \delta_G \geq \tau - \frac{N}{P^*}, \tag{23}$$

$$c_{G,M} = M - N - P^*\delta_G \leq M. \tag{24}$$

Proof. Notice first that fiscal creditors can always avoid bankruptcy by not trading, and find it strictly preferable to going broke, so any equilibrium is without bankruptcy. The proof takes five steps.

Step 1: $\Pi(\mathcal{P}) = \emptyset$ when $\delta_G \geq 0$. Suppose otherwise that there is an active trading post. There has to be an active private net seller since the state buys. At the lowest price at which there is a private net seller, this net seller does not buy at a higher price from Lemma 2, and cannot by definition buy at a lower price. She would thus be strictly better off reducing her order, a contradiction.

Suppose for the rest of the proof that $\delta_G < 0$. There is no equilibrium with no trade in this case as one agent could deviate and buy goods from the state.

Step 2: All predictable prices are weakly larger than P^*. There exists a "P^*-equilibrium" in which each private agent $i \in \mathcal{I}$ places a buy order for N_i/P^* units at P^*. Suppose there exists an equilibrium with active trading at another price. Let us denote $\underline{P}$ the lowest unofficial price. There has to be an active net seller at this price. She must be net buyer too otherwise she would be strictly better off cutting her order. She must buy below $\underline{P}$ from Lemma 2, and by definition cannot do so from a private seller, so she does so at $P^* < \underline{P}$.

Step 3: The P^*-equilibrium is unique when $N + P^*\delta_G < 0$. In this case buyers at P^* cannot be rationed since the private sector as a whole cannot bid more than N at P^* in an equilibrium without bankruptcy. Condition (15) implies that there cannot be a private net buyer at $\underline{P} > P^*$ defined above.

Step 4: Equilibrium with unofficial trade when $N + P^*\delta_G = 0$ or $N + P^*\delta_G > 0$ and $N_i \neq n$ for some $i \in \mathcal{I}$. We leave it to the reader to check that one can construct verbatim the equilibrium that we construct in the proof of Lemma 5 in the presence of fiscal debtors. The only case which slightly differs is that in which $N + P^*\delta_G = 0$ and all agents are ex ante identical. In this case, one equilibrium can be such that one of them buys at an unofficial price defined as in the proof of Lemma 5. The others sell all their goods at this price and reinvest the proceeds in the official market. Unlike when $N + P^*\delta_G > 0$ and agents are ex ante identical tackled in Step 5 below, this is an equilibrium as condition (13) is not necessary in the case in which $N + P^*\delta_G = 0$. Finally, that unofficial prices are strictly above $N/(-\delta_G)$ follows directly from Lemma 2.

Step 5: The P^*-equilibrium is unique when $N_i = N$ for all $i \in \mathcal{I}$ and $N + P^*\delta_G > 0$. Suppose by contradiction that there is unofficial active trade, and let $\bar{P}$ and (again) $\underline{P} > P^*$ respectively denote the largest and smallest unofficial active prices. Let i denote a net seller at $\underline{P}$ and j a net buyer at $\bar{P}$. The official market is rationed on the buy side ($d(P^*) < s(P^*)$), so that condition (13) applies to i.[6] Conditions (13) and (15) together imply $\Delta_i(P^*) \geq P^*/\underline{P} \geq P^*/\bar{P} \geq \Delta_j(P^*)$, requiring that i is unofficial net buyer above $\underline{P}$ or/and j is unofficial net seller below $\bar{P}$, either way a contradiction given Lemma 2. $\square$

Proposition 7 first states that in the presence of heterogeneous net transfers, the state must supply strictly more goods $-P^*\delta_G$ than the equilibrium average net demand N ($N < -P^*\delta_g$) in order to (strongly) determine the price level. Unlike in the situation with fiscal debtors in Proposition 6, the excess backing can be made arbitrarily small however, no matter the amount of heterogeneity across agents.

In the case of insufficient backing $N \geq P^*\delta_G$ that we deem "financial repression", heterogeneity among net transfers creates gains from trades among creditors. Whereas there still exists an equilibrium without unofficial trades, there also exist equilibria with unofficial prices strictly above $N/(-\delta_G)$. In these equilibria, small creditors are willing to acquire money at a low cost (at a high price level) in order to gain more dilution power in the official market. Conversely, agents with large claims accept to sell some money to them at such prices rather than further diluting their own positions in the official market. In other words, agents with low cash holdings arise as endogenous intermediaries between cash-rich agents and the state. We will show below that in the limit of negligible agents, unofficial trades vanish in the case in which $N + P^*\delta_G = 0$. They persist however when $N > -P^*\delta_G$.

Symmetry with debt deflation. Another way of describing such equilibria with multiple trading posts is that agents with little cash coordinate on squeezing the official market by bidding borrowed cash. This forces agents holding more cash to sell it at a high unofficial price level, thereby financing the squeezing strategy. These equilibria are thus symmetric to the debt-deflation ones in which agents with low fiscal debt corner the official market by flooding it with goods thereby forcing the more indebted ones to sell goods at a low unofficial price.

It is worthwhile noticing that in both cases, more unofficial trades yield more redistribution from the agents with the largest cash positions in absolute values towards the

[6] To see why, notice that if the official market is not rationed for buyers, then an unofficial one has to be, but then (15) implies that the active buyers must all have a strictly smaller footprint in this market than in the official one, which is impossible.

others, as the former force the latter to trade at unofficial prices that are less favorable than the official one.

Asymptotically atomistic economies. It is interesting to separate out, among the results in Proposition 7, the ones that survive in the limit in which each private agent becomes negligible. To be sure, the results that hinge on private agents' price impact are interesting in their own right, as there is ample evidence that the large institutions that participate in the primary markets for public liabilities have some price impact in practice. Yet assessing our results in the ideal case of negligible agents is also instructive. Here we show that when the economy converges to one in which each agent becomes negligible, the unofficial prices that may arise in the presence of financial repression all tend to $N/(-\delta_G) \geq P^*$.

Proposition 8. ***(Negligible agents)*** *Consider a sequence $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of policies with financial repression each associated with an economy of size n, and each such that the net transfers are not all identical. Suppose that $\mathcal{P}^n \to \mathcal{P}$ such that $-P^*\delta_G > 0$ and that*

$$\max_{i\in\mathcal{I}} \left\{ \frac{N_i^n}{nN^n} \right\} \underset{n\to+\infty}{\to} 0. \tag{25}$$

For every $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that for all $n \geq m$, the unofficial prices that are predictable given $\mathcal{P}^n$ belong to $(N/(-\delta_G) - \epsilon, N/(-\delta_G) + \epsilon)$.

Proof. Conditions (15) and (25) together imply that every unofficial price must become arbitrarily close $P^*/\delta(P^*) = N/(-\delta_G)$ as $n \to +\infty$. □

Condition (25) encodes that agents become negligible in the limit. Notice that this proposition implies in particular than when the state backs money with the exact real amount given P^*—$N + P^*\delta_G = 0$—then all equilibrium prices converge to the official target P^*. If $N + P^*\delta_G > 0$, there still are multiple equilibria with varying trading volume at unofficial prices, including possibly no unofficial trade. Yet all unofficial prices become arbitrarily close to $N/(-\delta_G)$.

The coefficient $\Delta_i(P^*)$ defined in (12) that drives agents' indifference between distinct prices illustrates the respective contributions of insufficient backing on one hand ($N + P^*\delta_G > 0$) and of the price impact of non-negligible agents on the other hand to the possible rise of unofficial trades. This coefficient is the product of the term

$(d(P^*) - d_i(P^*))/d(P^*)$ that reflects individual price impacts and vanishes as agents become negligible, and of $s(P^*)/d(P^*)$. That $-\delta_G = s(P^*) < d(P^*) = N/P^*$ in the case of financial repression is then the only remaining source of unofficial trade as agents become negligible.

Revisiting the case $N_- N_+ > 0$. It is easy to see that Propositions 6 and 7 together imply that if the state sells $-\bar{\delta}_G$ at $\bar{P}$ such that $N_+ + \bar{P}\bar{\delta}_G < 0$, and buys $\underline{\delta}_G$ at $\underline{P} < \bar{P}$ such that $N_i + \underline{P}\min\{\underline{\delta}_G, e - \tau\} \geq 0$ for all $i \in \mathcal{I}$, then the predictable prices must be within $[\underline{P}, \bar{P}]$, an interval that can be made arbitrarily small.

3.4 Feasible policies

We have thus far allowed for policies such that, depending on the strategy profile of the private sector, the state possibly ends up with strictly negative consumption of goods or/and money. Here we show that all the above results still hold for policies that we deem "feasible". These policies are such that the state's consumption is always positive no matter what the private sector does:

Definition 5. ***(Feasible policies)*** *A policy is feasible if and only if the state consumes positively goods and money for every private strategy profile.*

We have:

Proposition 9. ***(Characterization of feasible policies)*** *A policy is feasible if and only if*

$$\tau + \delta_G \geq 0, \tag{26}$$

$$M \geq N + P^* \min\left\{\delta_G^+, e - \tau\right\}. \tag{27}$$

Proof. We prove each inequality in turn.

Positive consumption of goods. The state transfers goods to the private sector only through sales, and not more than $-\delta_G$ per capita, ensuring that condition (26) is sufficient. Suppose that agent $i \in \mathcal{I}$ buys an arbitrarily small quantity at an arbitrarily large price from agent $j \in \mathcal{I}$ whom in turn bids the money, supposed to be larger than $-P^* n\delta_G$, in the official market. Other agents do not trade in the official market. Agent i also sells $e - \tau$ at an arbitrarily low price. Then the state must sell $n\delta_G$ units and receives arbitrarily few goods from the possible bankruptcy of i, establishing that (26) is also necessary.

Positive consumption of money. The right-hand side of condition (27) corresponds to the amount of money that the state must transfer to the private sector when the latter sells as many goods as possible and pays its taxes. This is the maximum amount M that the state needs to issue across all private profiles since the state issues additional money when agents default by assumption. □

Conditions (26) and (27) merely state that the state has enough real resources $n\tau$ and prints enough money nM to consume positively goods and money no matter the strategy profile of the private sector. Condition (26) ensures that the state consumes a positive quantity of goods no matter the private strategy profile. Condition (27) ensures that the state consumes a positive quantity of money for all private strategies.

It is important to stress that all the results up to Proposition 8 imply restrictions on all the components of a policy $\mathcal{P}$ except for M and τ. Thus all these results apply in particular to feasible policies, all that is needed is that M and τ be taken sufficiently large that the feasibility conditions (26) and (27) hold for the policies of interest. In particular, when price-level determination requires that the state stands ready to sell enough goods (a lower bound on $-\delta_G$), the associated feasible policies must feature sufficiently high real resources τ for the government.[7] When the state must be willing to sell enough money to fiscal debtors (a lower bound on δ_G), feasible policies must feature enough money creation M.

3.5 Defaultable security

The policy assumed thus far comprises a fixed nominal payment L_i from the state to private agent $i \in \mathcal{I}$. This section considers policies in which the payments to the private sector are subject to default, in the sense that the aggregate payment cannot exceed the aggregate amount of money that the state collects. In other words, the state does not

[7] Notice that a lower bound on τ is also one on $e \geq \tau$.

use the money that it creates to honor its transfers but only the one that it receives from the private sector.

We modify the baseline policy with only fiscal creditors studied in Proposition 7 as follows. For simplicity, we assume away cash taxes: $T_i = 0$ for all $i \in \mathcal{I}$. More important, we suppose that the positive cash transfer from the state to agent $i \in \mathcal{I}$ depends on the amount of money that the state collects in the market. In game-theoretic language, this transfer is no longer an action of the state but rather a (negative) payoff per capita $L_i(\mathcal{S})$ that depends on the private strategy profile $\mathcal{S}$. Formally, there exists $(B_i)_{i\in\mathcal{I}}$, a positive vector such that $B = \sum_{i\in\mathcal{I}} B_i/n > 0$, and such that the transfer to agent $i \in \mathcal{I}$ is

$$L_i(\mathcal{S}) = B_i \min\left\{1, \frac{\left(\sum_{i\in\mathcal{I}} P^*\hat{d}_i(P^*) - P^*\hat{s}_i(P^*)\right)^+}{nB}\right\}, \tag{28}$$

In case of default, each agent receives a share in the nominal value of the effective net demand at the official post. Definition (28) implies that creditors are treated pari passu. This is to fix ideas, all our insights carry over with alternative seniority rules.

The rest of the policy is unchanged and so are the trading and bankruptcy mechanisms. For brevity we consider only the case $\delta_G < 0$. We have:

Proposition 10. ***(Defaultable security and price-level determination)*** *The policy is feasible if and only if condition* (26) *holds. For all $D \in [(B + P^*\delta_G)^+, B]$, there exist equilibria in which the state pays $B - D$ per capita. Across such equilibria,*

$$c_{G,C} = \tau - \frac{B - D}{P^*},\ c_{G,M} = M. \tag{29}$$

There is no price-level determination. However, all active price levels converge to P^* as agents become negligible in the sense of Proposition 8.

Proof. Notice first that agents can always avoid bankruptcy by not trading and are strictly better off this way, so that any equilibrium must be without bankruptcy. Notice also that there exists a no-trade equilibrium, so that determination of the price level is at best weak. We leave it to the reader to check that for every $D \in [(B + P^*\delta_G)^+, B)$, there exists a P^*-equilibrium in which only the official trading post is active. Agent $i \in \mathcal{I}$ bids $B_i(1 - D/B)$ at P^* and collects the same transfer from the state. The rest of the proof is in three steps.

Step 1: Feasibility. By construction the state transfers only what it receives, and so it does not need to create M to consume money positively (but of course can always do so), that is, $c_{G,M} = M$ in and out of equilibrium.

Step 2: Unofficial prices tend to P^* as agents become negligible. Suppose that an equilibrium features active unofficial trading. Suppose that there is a net buyer at $P^b > P^*$. Lemma 3 applies with $P' = P^b$ and $P = P^*$. To see this, notice that the deviation used to prove the lemma—shifting part of the P'-bid towards P—strictly improves solvency as it increases the transfer received by the deviating agent. Thus condition (15) holds and P^b becomes arbitrarily close to P^* as agents become negligible. Suppose then that there is a net buyer below P^*, and let P^b denote the smallest price at which there is one. There must be net sellers at this price. Lemma 2 applies between P^b and any other unofficial price $P \neq P^*$, implying that these net sellers, who must be buying somewhere, buy at P^*. Suppose that such a net seller $i \in \mathcal{I}$ reduces her effective sales at P^b by $\epsilon > 0$ sufficiently small. Her order at P^* must then shrink by x such that

$$P^* x = \epsilon P^b + \frac{B_i P^* x}{nB}, \tag{30}$$

where the second term on the right-hand side reflects that her smaller bid reduces her net transfer from the state, and thus tightens her solvency constraint. Equilibrium requires that $x \geq \epsilon$, or $P^b \geq P^*[1 - B_i/(nB)]$, and so, as $P^b < P^*$, P^b must become arbitrarily close to P^* as agents become negligible.

Step 3: There exists equilibria with unofficial trade. For D such that $B - D + P^*\delta_G < 0$, let $i \in \mathcal{I}$ an agent with a minimum value of B_i— i need not be unique. Let this agent buy a sufficiently small quantity at $P = P^*[1 - B_i/(nB)]$ and all the others sell their $e - \tau$ at this post and invest the proceeds at P^*. It is an equilibrium as the unofficial buyer is indifferent between buying at P and P^* from the same reasoning as that used in step 2 and sellers may strictly prefer to sell more (unless their claim is equal to B_i in which case they are indifferent) but cannot. □

Within the limits of a static model, the equilibria are reminiscent of self-fulfilling debt crises. Equilibria with default here are gridlocks whereby the private sector does not bid much cash for goods because it expects sovereign default in the form of a small transfer, and these small bids in turn vindicate the small transfer. A key difference with the case of a nominally safe security is that there is no longer room for strict financial repression since the payment of the state by construction never exceeds the amount of money that it is willing to purchase at P^*. As a result, the situation bears similarities with that in which $N + P^*\delta_G = 0$ with safe securities: Unofficial trades are made possible only because of individual price impacts. Thus unofficial prices all become arbitrarily close to the official one when agents become negligible. Yet there are multiple equilibria with varying default severity even in this negligible limit, but only the goods consumption of the state varies across them. The price level does not. A higher loss given default creates additional real resources for the state.

More on out-of-equilibrium monetization. We find this result that the absence of any monetization of the transfer creates self-justified default to be remarkable. Yet it is important to stress that this is a knife-edge one. By committing to an arbitrarily small out-of-equilibrium financial repression, the state can eliminate all the equilibria but the one without default. To see this, suppose that for $\epsilon > 0$ arbitrarily small, one replaces (28) with

$$L_i(\mathcal{S}) = B_i \min\left\{1, \frac{\left(\sum_{i\in\mathcal{I}} P^*\hat{d}_i(P^*) - P^*\hat{s}_i(P^*)\right)^+}{nB} + \epsilon\right\}, \tag{31}$$

which requires a money creation of ϵ to be feasible in the sense of Proposition 9. Then it cannot be that an agent receives less than B_i in equilibrium as this would imply that the private sector does not bid its aggregate cash holdings in the official market, which is inconsistent with the rationality of at least one private agent.

3.6 Market-clearing policy

The goal of this section is to compare the outcomes when the state's trading strategy consists in posting a fixed price as above with those when the state acts as an auctioneer à la Shapley and Shubik (1977), setting a price that absorbs all the private money supply in equilibrium. This will highlight the crucial role of the trading protocol on the set of predictable price levels.

We modify again the baseline policy with fiscal creditors only as follows. First, for brevity, we restrict again the analysis to policies such that $T_i = 0$ for all $i \in \mathcal{I}$ and $\delta_G < 0$.

Second, the official trading post no longer operates as the unofficial ones, but rather as a "sell-all" market à la Shapley and Shubik (1977). Private agent $i \in \mathcal{I}$ bids a positive quantity of money $C_i \geq 0$. The official price is then a function of the private strategy profile $\mathcal{S}$ defined as

$$P(\mathcal{S}) \equiv \frac{\sum_{i\in\mathcal{I}} C_i}{-n\delta_G}. \tag{32}$$

Finally, since our main goal is to highlight how defaultability and the trading protocol jointly determine the price level or fail to do so, we posit that the transfers to the private sector feature both a safe and a defaultable component. There exists a positive sequence $(l_i, B_i)_{i\in\mathcal{I}}$ such that $l = \sum_{i\in\mathcal{I}} l_i/n > 0$ and that the net transfer to agent $i \in \mathcal{I}$ is

$$L_i(\mathcal{S}) = l_i + \frac{B_i}{B} \min\left\{B, \left(\frac{1}{n}\sum_{i\in\mathcal{I}} C_i - l\right)^+\right\}, \tag{33}$$

where $B = \sum_{i\in\mathcal{I}} B_i/n$.

Proposition 11. ***(Market-clearing policy and price-level determination)*** *The policy is feasible if and only if condition* (26) *holds and* $M \geq l$*. The set of predictable price levels includes* $[-l/\delta_G, -(l+B)/\delta_G]$*. For every* $D \in [0, B]$*, there exists an equilibrium with trades only in the official market at the price* $-(l + B - D)/\delta_G$*, and there exist equilibria in which such official trades coexist with active unofficial markets. Furthermore, across all equilibria* $c_{G,C} = \tau + \delta_G,\ c_{G,M} = M$*.*

The set $\Pi(\mathcal{P})\backslash[-l/\delta_G, -(l + B)/\delta_G]$ *becomes negligible as agents become negligible in the sense of Proposition 8.*

Proof. By construction the state transfers only what it receives beyond nl and positive consumption of money thus only requires $M \geq l$. Notice that there is always trade in equilibrium as $l > 0$ and $\delta_G < 0$. We leave it to the reader to check that for every $D \in [0, B]$, there exists an equilibrium whereby each agent $i \in \mathcal{I}$ bids $C_i = l_i + B_i(1 - D/B)$ at the official post and the price is $-(l + B - D)/\delta_G$. The proof that unofficial prices converge to the official one when agents become negligible is similar to that in Proposition 10, and so we omit it. All that is left to establish the proposition is the construction of an equilibrium with unofficial trade.

Equilibria with unofficial trade. One can construct an equilibrium with a high unofficial price level similar to those when the official price is fixed and $N + P^*\delta_G = 0$. The reason is that an agent who buys dear has the same payoff from deviating towards the official post with a market-clearing price as with a fixed official price and uniform rationing. Formally, consider some agent $i \in \mathcal{I}$ and $0 < C_i < l_i + B_i(1 - D/B)$. Let us construct an equilibrium in which agent i posts C_i on the official market and sells a nominal value $l_i + B_i(1 - D/B) - C_i$ in an unofficial market with price P defined below. All the other agents post all their goods on the unofficial market and are uniformly rationed. They also bid a nominal amount $C_{-i} = \sum_{k \in \mathcal{I}} l_k + B_k(1 - D/B) - C_i$ in the official market. Let us denote $P(x) \equiv (C_{-i} + C_i + x)/(-n\delta_G)$ the price on the official market when agent i bids $x + C_i$ in the official market. The trading strategy for all agents except i is optimal whenever $P > P(0)$ and posting C_i on the official market is optimal for agent i when $0 = \arg\max (C_i + x)/P(x) + (l_i + B_i(1 + \min\{x; D\}/B - D/B) - C_i - x)/P$. The term $\min\{x; D\}B_i/B$ stems from the fact that bidding an additional x on the official market leads to an additional xB_i/B units of money for agent i as debt repayment. When $D > 0$, when $C_{-i}/(C_{-i} + C_i) = (1 - B_i/B)P(0)/P$, the global optimum is $x = 0$. Otherwise, when $D = 0$, this condition is $C_{-i}/(C_{-i} + C_i) = P(0)/P$. As these two conditions may hold jointly, this proves the existence of an unofficial market with exchange $l_i + B_i(1 - D/B) - C_i$ of money at the price $P = (C_{-i} + C_i)^2/(-n\delta_G C_{-i})$. □

An implication of Proposition 11 is that a market-clearing policy does not prevent the emergence of unofficial markets, at least away from the limit of negligible agents. When some agents bid more money than what they initially have on the official market by buying money on unofficial markets, they can force at least one agent to sell part of her money at a higher price level. This higher price level makes this latter agent indifferent between this unofficial trade and crowding herself out in the official market.

The change in trading protocol flips the relationship between default and price-level determination. The salient implication of Proposition 11 is that shifting the official trading protocol from fixed price to fixed quantity flips the relationship between the defaultable nature of public liabilities and price-level determination in the negligible limit. Proposition 7 shows that the fixed-price trading protocol may fail to determine the price level in the presence of a non-defaultable security because financial repression opens up the possibility of trade at multiple prices in equilibrium. Proposition 10 shows that by contrast, this fixed-price protocol determines the price level in the presence of a defaultable security in the negligible limit because only spending varies across equilibria with varying haircuts on public debt. The market-clearing trading protocol generates the exact opposite prediction. The price level is determined in the negligible limit if and only if the security is non-defaultable. In the presence of defaultable securities, it is the price level that absorbs all the fluctuations in the haircut on public debt across equilibria with varying default severity, whereas state spending remains unaffected.

In sum, in the negligible limit, safe securities warrant price-level determination in the presence of a market-clearing official price whereas defaultable ones do so when the official price is fixed.

4 Applications

This section discusses several applications of our framework.

Fiscal backing: Assignats during the French Revolution. During the French Revolution, the state issued paper money in 1790, "Assignats", to reimburse debts (N) and at the same time was selling the National Estates ($-\delta^G > 0$) constituted by assets seized from the church. Sargent and Velde (1995) describe the "rise and fall of the assignat" as a sequence of three periods: a "real-bills" period, a "legal restriction" period and an "hyperinflation" period. We can connect each of these periods to specific aspects of our model.

In the "real bills" period, the value of National Estates was about 2,400 millions livres and exceeded the value of the debt that the state had to honor, which was around 2,000 millions livres. In the terms of our model, in nominal terms, $N + P^*\delta^G > 0$: assignats were then backed. Here, P^* is the face value of assignats.

During the next two periods, a large quantity of assignats were issued to compensate for limited fiscal resources due to war and internal chaos. We interpret these two periods as situations in which $N + P^*\delta^G < 0$: the state does not back its currency. However, we note important differences between the two periods.

During the "legal restriction" period, under the Terror, the state implemented very harsh restrictions on hoarding assets, closed markets, and imposed wage and price controls. As Sargent and Velde (1995) note, these restrictions led to a "guillotine-backed currency":

> Under the Terror, any citizen accused of violating these laws could expect swift and arbitrary proceedings. The law on parity of the assignat called for arraignment and trial within 48 hours of the offense. The law encouraged denunciations from informants and gave extravagant powers to local authorities to enforce the restrictions. In a few dozen instances, the death penalty was imposed for crimes against the assignat or for hoarding.

The outcome is that the price level was determined and no inflation arose, consistently with our model in which the absence of private trades leads to price determination with financial repression in the case $N + P^{*}\delta^{G} < 0$.

With military successes, the Terror was overthrown and the legal restrictions were alleviated and markets reopened. In this "hyperinflation" period, private agents tried to sell all their holdings of assignats for goods and specie. This period opened up arbitrage opportunities. As reported by Sargent and Velde (1995):

> In terms of gold, prices were lower than in 1790, creating trading opportunities for the savvy. A Swiss visiting Paris hastened to change his gold for paper; bought hundreds of shoes, stockings, and hats; shipped them off to Switzerland; and lived in Paris like a king for a month.

We find interesting that this anecdote would not hold with Walrasian markets but requires some arbitrage opportunity. The interpretation consistent with our model is that these arbitrage opportunities stem from the rationing on the demand side of the market between gold (or specie) and currency.[8] This kind of intermediation looks very much like the one we obtain in our model, where at least one agent may find optimal to sell its goods against money because of rationing on the post of the state.

[8]To be precise, this market should be more rationed than the one between currency and shoes, stockings, and hats, at least so for the "Swiss", who can export the goods to Switzerland, while locals have perhaps already more than satiated their demands for these goods.

Exchange rate pegs and parallel market exchange rates. Parallel foreign exchange markets may exist for multiple reasons (e.g., in order to evade capital controls or for illegal transactions). They may also emerge from countries maintaining an overvalued official exchange rate and rationing the supply of foreign currency. In particular, Gray (2021) notes that, among the motives for such a policy decision:

> It may also be promoted by those who can profit from privileged access to FX at the official exchange rate (rent seeking behavior).

This echoes the motive for the emergence of parallel markets in our model: they emerge from the heterogeneity of agents and their price impacts on the official market. Gray (2021) also provides a list of 19 countries with official and parallel markets in the 2010-2020 period. More generally, by using estimates of export misinvoicing practices,[9] Reinhart and Rogoff (2004) show that parallel markets are widespread in fixed exchange rate regimes and concern more than half the pegs, since at least World War II. As they show, parallel market exchange rates are better indicator of monetary policy stance and they even predict realignments in the official exchange rate. This aspect of parallel markets connects with our results that the price on parallel markets may reflect the actual backing of currency $N/(-\delta_G)$.

Price controls. Another interpretation of our model is price controls. With price controls, the government imposes the prices at which private agents trade goods against money. Imposing a price below the equilibrium one may lead to rationing. Such a rationing may be reinforced by lower production by firms facing lower prices. In this case, the post of the government can be interpreted as the supply of goods $-\delta_G > 0$ supplied by firms at the controlled price P^*. Notice that, with this interpretation, the feasibility conditions described in Section 3.4 do not apply, as the supply of goods $-\delta_G$ does not stem from the taxation power of the government.[10]

The connection between price controls and parallel markets, or, for instance, black markets, is well documented, and examples of black markets associated with price controls abound: the US during World War II (e.g., Rockoff, 2004), Argentina in 1973-1975 (e.g.,

[9] According to IMF (1991), the ways to circumvent official markets are "smuggling, over-invoicing of imports and under-invoicing of exports, workers' remittances from abroad, and tourism".

[10] A full model of price controls would include the endogenous supply of goods by firms $-\delta_G(P) > 0$ as a function of the controlled price P, with all the potential tools that the government may have to force firm production.

Chu and Feltenstein, 1978), Chile in 1973 (e.g., Edwards, 2023) are famous examples. This connection is so well established that the Wikipedia page on black markets even mentions "Common motives for operating in black markets are to trade contraband, avoid taxes and regulations, or evade price controls or rationing." Notice that, in our model, black markets take place between net buyers of goods (net sellers of money). With price controls, another important dimension of black markets is that they take place between net sellers of goods (net purchasers of money) and net buyers of goods (net sellers of money): firms may participate themselves in black markets to sell their production at better prices compared with the controlled one.

In-kind taxation in periods of financial repressions. Condition (22) relates the real value of state liabilities to the real surplus in the absence of financial repression as follows:

$$\frac{1}{P^*}\sum_{i\in\mathcal{I}} L_i = f - nc_{G,C}, \tag{34}$$

where $f \equiv n\tau + (\sum_{i\in\mathcal{I}} T_i)/P^*$ are the real fiscal resources of the state. Thus, among the policies that determine the price level, those that differ only along the modalities of tax payment—in-kind versus in cash—but not along the real value of taxes f lead to the same real allocations. By contrast, the modalities of tax payment are no longer irrelevant under financial repression. As is transparent from expressions (23) and (24), a reduction in $\sum_{i\in\mathcal{I}} T_i$ and increase in τ holding f fixed shifts real resources from the private sector towards the state since this reduces the private demand for money for tax-payment motives, and thus increases households' forced money holdings. This is reminiscent of historical situations in which a financially distressed public authority imposed in-kind payments for some taxes, such as the Confederacy during the US civil war or the USSR in the 20s.

More precisely, the Confederacy faced important difficulties to levy taxes and had to rely on money to finance its war efforts. Through the lens of our model, the fraction of new money used to pay for the wages of the army, for example, can be understood as nominal transfers N. The large issuance of money led to hyperinflation. In particular, no fiscal backing was supporting the value of money: As noted by Nielsen (2005), "The Treasury bills issued during the war had a peculiar feature: They were redeemable for gold two years after the war ended, which meant that the value of the bills was partially tied to expectations of victory for the Confederacy." This lack of backing can be understood as a situation of financial repression in which $N + \delta_G P^* > 0$.

Consistently with this situation of financial repression, the mix between nominal and in-kind taxes did matter for the Confederacy: As documented by Burdekin and K. (1993), in-kind taxes contributed more than 50% of the Confederate revenue for example in the ten first months of 1863.

Private monies. While our main set of implications concerns government-issued liabilities, our framework may also be used to think about privately-issued monies.

Stablecoins. Our model provides a useful framework for analyzing the determination of stablecoin prices. Major stablecoins aim to maintain a peg to the US dollar by backing tokens with USD-denominated assets, such as US Treasury bonds. Typically, issuers exchange tokens for dollars in a primary market and promise to redeem them at par value. Stablecoins are then traded in secondary markets.[11] In the context of our model, the stablecoin represents "money", while the US dollar is the "good". On the primary market, the issuer (analogous to the state in our model) posts a sell order $-\bar{\delta}_G$ at price $\bar{P}$ and a buy order $\underline{\delta}_G$ at price $\underline{P}$, with the spread $\bar{P} - \underline{P}$ representing transaction fees. The feasibility conditions involve the dollar value of reserve assets held by the stablecoin issuer. According to our model, stablecoin prices on the secondary market should fluctuate within this band as long as token issuance $\underline{\delta}_G$ exceeds all the potential demand ($\underline{\delta}_G \geq \max_{i \in \mathcal{I}}(-N_i)^+/\underline{P}$) and the backing remains sufficient, i.e., $-\bar{\delta}_G > N_+/\bar{P}$. If the backing is perceived as insufficient, prices may breach the upper bound $\bar{P}$, as seen during the brief depeg of USDC following the SVB failure in March 2023 (Aldasoro et al., 2023).

While fiat-backed stablecoins have been the most successful, they are not the only type. Before the Terra Luna collapse in May 2022, algorithmic stablecoins were gaining popularity.[12] In this ecosystem, the stablecoin UST was circulating and differed from fiat-backed stablecoins in its peg mechanism. A smart contract allowed the exchange of one UST for $1 worth of LUNA —the native token circulating on Terra and a claim

[11] Numerous studies have explored stablecoin stability: Lyons and Viswanath-Natraj (2023), d'Avernas et al. (2022), and Routledge and Zetlin-Jones (2022), among others.

[12] The Terra Luna ecosystem was the third largest in the market, with a capitalization of $50 billion, which collapsed to nearly zero in just three days.

on Terra's transaction fees and block rewards, but also a token giving access to Terra applications. This setup resembles the primary market described earlier, except that δ_G was not fixed in USD but varied with LUNA's dollar value, which depended on the overall value of the Terra system. The collapse of UST was triggered by a sharp decline in LUNA's price, stemming from a sudden loss of confidence in the system's sustainability (see Liu et al., 2023, for further details). Liu et al. (2023) highlight heterogeneity in the use of the primary versus secondary markets during the May 2022 crisis:

> Interestingly, we find that Alameda Research, a cryptocurrency trading firm closely affiliated with the FTX exchange, conducted the largest amount of UST-LUNA swaps among Anchor depositors. It seems that the swap fees and uncertainty about the execution price of LUNA on exchanges discouraged most other Anchor depositors from utilizing the native swap contract as an exit strategy. But Alameda Research, with its advantageous access to the FTX exchange, had a competitive advantage over other market participants.

This observation suggests that certain sophisticated agents, such as Alameda Research, had a competitive edge in trading on the primary market due to their superior access and resources to resell LUNA, while other participants opted for the secondary market, even at significantly discounted prices. In this scenario, market segmentation arose not from differences in price impacts due to rationing and order sizes (as modelled in our setting), but from the varying capabilities of investors.

Another noteworthy aspect of the crash was the role of Anchor, a lending platform for UST. TerraForm Labs, the creator of the Terra network, subsidized the interest rate on UST deposits in Anchor to stimulate demand. Until May 2022, the interest rate stood at 19.5%. This high rate, coupled with short-term price stability followed by rapid price fall, mirrors item (iii) of Proposition 13.

Free Banking era. Under Free Banking, as experienced by the US between 1837 and 1863, banks' liabilities were in form of banknotes redeemable in specie. These banknotes, in contrast with deposits,[13] were massively exchanged on secondary markets. In particular, brokers specialized in trading these banknotes. As Gorton (1999) showed (see

[13] As noted by Gorton (1985), deposits are "double claims" that is 'a claim on a specific agent's account at a specific bank.

also Jaremski, 2011), the prices at which these banknotes traded reflected banks' default risk, which relates to the option to redeem the banknote (and the transportation cost to travel to the bank to redeem it) – in particular, "wildcat" banking, i.e., banks which intentionally overissue money compared with their ability to redeem it, is considered to be at most marginal. To be more precise, Jaremski (2011) notes that banknotes were traded at par locally, unless the bank was closed or suspended. From the perspective of our model, money is banknotes while the good stands for species. The commitment of the bank to redeem banknotes at par is consistent with a fixed-price order. The situation in which there is no redemption risk is akin to one in which backing is sufficient ($\delta^G P^* + N < 0$), while the one in which banknotes are traded at a discount, even in their place of issuance, is one in which backing is insufficient, ($\delta^G P^* + N > 0$).

Money Market Funds and other Asset-backed funds. Another application of our model could be Money Market Mutual Funds. These funds implicitly guarantee that the value of their shares is 1\$. Interpreting the government in our model as the MMF and the private sector as the MMF's shareholders willing to withdraw their funds from the MMF, this guarantee can be modeled by a fixed-price order and $-\delta_G$ is the measure of resources that the MMF can use to redeem its shares. When this backing is sufficient ($N + P^*\delta_G < 0$), the price of shares is pegged at P^*. When this backing is insufficient ($N + P^*\delta_G > 0$), the price of shares may fall on secondary markets. Such situations of insufficient backing can be connected to the runs that MMF experienced in the aftermath of the 2008 financial crisis (see Gorton and Metrick, 2010, among many others).

In the case where the fund redeems a function of the value of its resources $-\delta_G$, the situation is then akin to the market-clearing price case that we discuss in Section 3.6.

5 Getting the Walrasian approach right

We mentioned in the introduction the two reasons we study price-level determination in a strategically closed environment instead of using a more standard Walrasian one. First, Walrasian markets introduce *de facto* trading bans that contribute to the determination of the price level, and we did not want to impose them on private agents nor on the state. Second, the Walrasian equilibrium concept makes it impossible to properly define feasible policies—policies that are physically possible no matter the actions of the private sector. This section applies the insights from our strategic model to analyze price determination in a Walrasian version of it. We perform a translation exercise: For each of the main policies studied in our strategic setup, we design a Walrasian counterpart that delivers the same outcomes as the ones we obtain in the limit of strategic but negligible agents. This exercise will make clear that studying price-level determination in a Walrasian framework leaves many essential dimensions of public financial policy under the rug, bypassing central and practically important issues of feasibility and out-of-equilibrium behavior. More specifically, we deliver the four following insights.

Insight #1: The fiscal theory of the price level is an extreme form of fiscal dominance. For notational simplicity, we study a version of our economy populated by $n = 1$ private agent—one needs only one representative agent in a Walrasian environment. Consider a policy comprised of a nominal transfer to that agent L, an in-kind tax τ levied on her, and government consumption of the good $\tau - \sigma$, all strictly positive real numbers. A Walrasian equilibrium associated with this policy (L, τ, σ) is comprised of consumption of money C_M and goods C_C by the private agent and of a price level $P > 0$ such that the price-taking private agent optimally consumes:

$$(C_M, C_C) = \arg\max C_C \tag{35}$$

$$s.t.\ C_M + PC_C + P\tau \leq Pe + L, \tag{36}$$

$$C_M,\ C_C \geq 0, \tag{37}$$

and, from Walras' law, such that the market for money clears:

$$L - C_M = P\sigma. \tag{38}$$

Condition (38) states that the private supply of money (endowment minus consumption of money) equals the state's demand of money (the nominal value of its surplus). Individual rationality requires that (36) binds and $C_M = 0$. Injecting this in the market-clearing condition (38) yields a unique equilibrium price level P associated with (L, τ, σ) that solves $L = P\sigma$: This is the so-called fiscal theory of the price level. In our setup, the case $B = 0$ in Proposition 11 corresponds (in the negligible limit) to such a policy with a fixed nominal liability and a fixed real quantity sold by the state for money. This policy is associated with two important assumptions. First, the state creates money and stands ready to use it to make good on its liability if it does not collect enough money in the market. We highlight that the state must stand ready to fully monetize the entire value of L this way in the (out-of-equilibrium) event that it does not collect any money in the official market. Second, the state adjusts the official price level in response to private demand so that its real surplus and in turn its consumption remain constant. In other words, our full-fledged model shows that an extreme form of fiscal dominance underlies the fiscal theory of the price level. The state prints money as needed to make sure that its liabilities are all perfect substitutes with money, standing ready to monetize all of it, and manipulates the price level so as to insure the government's consumption from the fluctuations of private demand. We now relax these assumptions and describe the Walrasian policy that generates the same outcomes as the strategic one in the absence of these assumptions.

Insight #2: Defaultable debt is real debt. We first relax the assumption that the transfer is nominally safe, while sticking to the one that the state trades at a market-clearing price. This corresponds to the other polar case in (the negligible limit in) Proposition 11 in which $l = 0$ and $B > 0$. In this case, our setup predicts that there is price-level indetermination. All price levels within $[0, -B/\delta_G]$ can be sustained as equilibria with increasing levels of default and deflation. In the Walrasian setup, it is straightforward to see that these outcomes obtain when the policy (L, τ, σ) becomes $(\min\{L, P\sigma\}, \tau, \sigma)$. In words, the transfer is contingent on the equilibrium outcome as it depends on P. As in the strategic case, all price levels within $[0, L/\sigma]$ are then equilibrium outcomes. This basically shows that when debt is defaultable in our setup in the sense that it is not monetized, it is as if it was real in the Walrasian environment (up to an upper bound). As is well-known, the fiscal theory of the price level does not hold if both debt and fiscal surpluses are real.

Insight #3: An indexed surplus and defaultable debt make a hidden peg. The situation in which the two implicit assumptions in the fiscal theory of the price level—debt monetization and market-clearing price—are relaxed is covered in (the negligible limit of) Proposition 10, in which debt is defaultable and the state trades at a fixed official price. In this case, in the negligible limit, the price level is determined at P^* but there are a continuum of equilibria with varying default and surpluses. The more the state repays the less it consumes. This situation can be viewed as one of monetary dominance since there is no money creation to honor public liabilities, and policy sets a fixed price level no matter the consequences for public consumption. It is again possible to obtain these equilibria in the Walrasian environment by defining an appropriate contingent policy. The policy has to be contingent on two equilibrium outcomes, the real value of private money supply that we denote s, and the price level P. The policy that generates the outcomes in Proposition 10 is $(\min\{L, Ps\}, \tau, P^*s/P)$. In words, holding the real value of private money supply s fixed, the state makes its real surplus decreasing in the price level so that the only equilibrium price is P^*. Since the state cannot explicitly set an official price level in the Walrasian environment, it uses a contingent surplus that eliminates all possible price levels but its target P^*. A potential connection with this strategy is the approach followed by Levy-Yeyati and Sturzenegger (2003) or Reinhart and Rogoff (2004) and the literature thereafter to identify "hidden pegs": these pegs correspond to a low variability in the exchange rates and a large one in the amount of FX reserves.

Insight #4: The Walrasian auctioneer bans the state from partially controlling the price level with financial repression. A final situation that is interesting to bring to the Walrasian environment is that of financial repression—$N > -P^*\delta_G > 0$ in Proposition 7. In the negligible limit, there are multiple equilibria in which the trading volume is split between the official market at P^* and the unofficial one at $-N/\delta_G$. The unofficial volume has an upper bound that is decreasing when agents have more homogeneous money holdings. A situation with two rationed markets for the same good is of course out-of-reach of Walrasian environments. The Walrasian model prevents financial repression, and predicts a price-level determination with a unique equilibrium price equal to the unofficial one under financial repression. Yet, unsustainable pegs with unofficial markets are commonplace in practice.

6 Two-date model

This section studies a simple extension of our model with two dates $\{0, 1\}$. The economy is still populated by a state and by $n \geq 2$ private agents. These agents value only a date-1 consumption good that is obtained out of the storage of a date-0 consumption good at a linear rate $\rho > 0$ between 0 and 1. Each agent $i \in \mathcal{I}$ is endowed with $e_i > 0$ units of the date-0 consumption good, such that (without loss of generality) $(e_i)_{i\in\mathcal{I}}$ is increasing. We denote $e = 1/n \sum_{i\in\mathcal{I}} e_i$. We suppose that there exists $i \in \mathcal{I}$ such that $e_i \neq e$. Our focus is on the connection (or lack thereof) between date-1 and date-0 price-level determination. For brevity, we restrict the analysis to the simplest policies that shed light on this question.

Policy. A policy $\mathcal{P} = (\delta_{G,0}, P_0^*, R, \delta_{G,1}, P_1^*)$ consists in two trades and one contingent transfer:

- **Trades.** The state stands ready to buy up to $n\delta_{G,0} > 0$ units of the date-0 good at a price P_0^*, and to sell up to $-n\delta_{G,1} > 0$ units of the date-1 good at a price P_1^*.
- **Transfers.** The state multiplies any outstanding net position in money by a private agent at the end of date 0 by $R > 0$, and this defines her net position at the outset of date 1.

Notice that, except for the fact that the interest rate applies to quantities that have been decided at date 0, we study only policies that are not contingent at date 1 on the date-0 actions of the private sector.

Private trades and bankruptcy. At each date $t \in \{0, 1\}$, private agents can submit any number of buy or sell orders of the date-t good, with the restriction that they cannot place sell orders for a total quantity larger than their endowment at the outset of each date. Trading posts clear with uniform rationing as in the one-date model.

With a straightforward extension of the one-date notations, the strategy of agent $i \in \mathcal{I}$ is $\mathcal{S}_i = (D_{0,i}(.), S_{0,i}(.), D_{1,i}(.), S_{1,i}(.))$. While it does not show in notations for parsimony, the date-1 orders are conditional on history, that is, on date-0 actions. Agent $i \in \mathcal{I}$ is bankrupt at date 0 if and only if

$$\int PdD_{0,i}(P) > \frac{1}{R}\left(\int Pd\hat{S}_{1,i}(P) - \int Pd\hat{D}_{1,i}(P)\right) + \int Pd\hat{S}_{0,i}(P), \tag{39}$$

and at date 1 if and only if

$$\int PdD_{1,i}(P) > R\left(\int Pd\hat{S}_{0,i}(P) - \int Pd\hat{D}_{0,i}(P)\right) + \int Pd\hat{S}_{1,i}(P). \tag{40}$$

Equilibrium concept. A profile $\mathcal{S} = (\mathcal{S}_i)_{i \in \mathcal{I}}$ is a predictable outcome given $\mathcal{P}$ if and only if it is a subgame-perfect Nash equilibrium.

This setup departs from the one-date model in two interesting dimensions. First, apart from interest payments, there are no transfers of any sign at any date imposed on private agents. Their date-1 net cash positions result only from their voluntary date-0 trades. Lemma 4 shows that this precludes any trade in the one-date model, we will see that it is no longer the case here. Second, money may deliver consumption at date 1, and thus serves as a store of value. In this sense, date 0 is an extension of the one-date model in which money may be intrisically desirable.

We present our results in two steps. The following proposition first characterizes the set of date-0 predictable price levels. We will then focus on situations in which the price level is determined at date 0 and study whether it is also the case or not at date 1.

Proposition 12. ***(Date-0 outcomes)*** *Let*

$$r^* \equiv \frac{RP_0^*}{P_1^*}.$$

- *If $r^* = \rho$, the date-0 price level is determined, equal to P_0^*.*
- *If $r^* > \rho$, the date-0 price level is strongly determined and equal to P_0^* if $e \leq \delta_{G,0}$. If $-\delta_{G,1}/r^* > e > \delta_{G,0}$, then it is not determined as there are equilibria with unofficial trade at prices below P_0^*.*
- *If $r^* < \rho$, the date-0 price level is not determined, there are equilibria with trades at unofficial prices. The set of predictable unofficial prices is $[P_0^*, \rho/r^* P_0^*)$.*

Proof. The proof is in three steps.

Step 1: There cannot be active trade above P_0^* if $r^* \geq \rho$. Suppose that $r^* \geq \rho$. Let $\bar{P}_0$ the largest price at which there is active trading. We show that $\bar{P}_0 \leq P_0^*$. Suppose otherwise. Net buyers at $\bar{P}_0$ do not fund their purchases from selling at a lower date-0 price as this would be strictly suboptimal. Thus they must do so by selling at date 1 at a price $P_1 \geq R\bar{P}_0/\rho > r^* P_1^*/\rho \geq P_1^*$. Only net sellers at $\bar{P}_0$ are willing to be their counterparts at date 1 because other date-0 net sellers must buy at strictly lower prices at date 1 to find it optimal. But then the net buyers at $\bar{P}_0$ cannot recoup enough cash at date 1 because the net (date-0) sellers must place some buy orders strictly below P_1 at date 1 from Lemma 3, a contradiction.

Step 2: Second bullet point in the proposition. Notice that if $r^* > \rho$, then no trade at date 0 is not an equilibrium: One agent could deviate and save at the official post. If $\delta_{G,0} \geq e$ selling below P_0^* is strictly dominated by selling at P_0^*. Suppose $\delta_{G,0} < e$ but $r^* = \rho$. We show that nor can there be active trade below P_0^* in this case. Suppose otherwise and let P_0 be the smallest active post. Net buyers and sellers at P_0 must take opposite sides at date 1 at $P_1 = RP_0/\rho < P_1^*$. But then P_0 sellers would be strictly better off selling at P_0^* and cutting their date-1 order so as to remain solvent if needed. Finally, suppose $r^* > \rho$ and $\delta_{G,0} < e < -\delta_{G,1}/r^*$. We construct an equilibrium with unofficial trade at a price below P_0^* as follows. For $e' \in (0, e_n)$, suppose that all agents sell their entire endowment at P_0^* but agent n who sells all of it except for an amount e' sold instead at a price $P_0 < P_0^*$. The other agents bid the (effective) proceeds from their P_0^*-sales at this P_0-post. Agent n's payoff is

$$\frac{n(e_n - e')}{ne - e'}\delta_{G,0}r^* + \left(e_n - e' - \frac{n(e_n - e')}{ne - e'}\delta_{G,0}\right)\rho + \frac{e'P_0r^*}{P_0^*}, \tag{41}$$

and the f.o.c. w.r.t. e' yields

$$P_0 = \frac{P_0^*}{r^*}\left[\rho + (r^* - \rho)\frac{n(ne - e_n)}{(ne - e')^2}\delta_{G,0}\right]. \tag{42}$$

A pair (e', P_0) satisfying this condition is an equilibrium since agent n invests optimally and the others would like to scale up their trades but have a binding constraint at the official post.

Step 3: Last bullet point in the proposition. Suppose finally that $r^* < \rho$. Let $P_0 \in [P_0^*, \rho P_0^*/r^*)$. An agent $i \in \mathcal{I}$ buying some goods at P_0 from an agent $j \in \mathcal{I}$ at date 0, storing them and then selling them back to j at $P_1 = RP_0/\rho = (r^*/\rho)(P_0/P_0^*)P_1^* < P_1^*$ is an equilibrium. That there cannot be active trade at prices outside $[P_0^*, \rho/r^* P_0^*]$ follows from the same arguments as the absence of unofficial trade above P_0^* when $r^* \geq \rho$ and below P_0^* when $r^* = \rho$. □

Proposition 12 first shows that a policy that features only trades may determine the price level. This contrasts with the one-date model (see Lemma 4). This also contrasts with the findings in Niepelt (2004) that the fiscal theory of the price level holds only in the presence of legacy liabilities in a Walrasian environment: in the sense that newly issued government debt does not determine the price level in economies that start without debt at all. We obtain this result both because money is a valuable store of value and because the state is price setter.

Another interesting result is that when the state does not offer a sufficient return on money relative to storage ($r^* < \rho$), agents may enter into trading sequences that emulate inside money. Some agents buy at a higher date-0 price than the official one P_0^* and store the purchased goods, financing the trade with a date-1 sale of the storage output at a lower price than P_1^* at date 1. That $r^* < \rho$ implies that these agents can manufacture this way the same flows as that of a nominal debt contract with real return ρ, and their counterparts find this therefore equivalent to storage (and superior to money).

When money offers a better return than storage, the price level is pegged at the official price as soon as the state is ready to accept as many goods as the agents are willing to sell at this price. Notice that we prove this result without using any characterization of date-1 outcomes. We now show that the price level may be determined this way at date 0 even when financial repression precludes determination at date 1.

Proposition 13. ***(Date-0 determination with or without date-1 determination)*** *Suppose $r^*[1 - e_i/(ne)] > \rho$ for all $i \in \mathcal{I}$ and $\delta_{G,0} \geq e$.*

(i) *The price level is strongly determined at date 0, equal to P_0^*.*

(ii) *If $-\delta_{G,1} > 0$ is sufficiently large or sufficiently small other things being equal, the date-1 price level is strongly determined, equal to P_1^*.*

(iii) *If $r^* e > -\delta_{G,1} > \rho/(1 - 1/n) \min_{i \in \mathcal{I}} e_i$, there is indetermination of the price level at date 1 with possible unofficial trades above P_1^*.*

Proof. Point i) is a direct consequence from Proposition 12. Regarding point ii), if $-\delta_{G,1}$ is sufficiently large that $r^* e + \delta_{G,1} < 0$, all agents optimally invest their entire (heterogeneous) endowments at the official post at date 0 and their savings are fully backed at date 1, ensuring price-level determination from Proposition 7. If $-\delta_{G,1}$ is sufficiently small, all agents invest the same amount $x > 0$ such that $-\delta_{G,1}(1 - 1/n)/x = \rho$ at date 0. The condition for this to happen is that $x \leq \min_{i \in \mathcal{I}} e_i$. At date 1 they are in the situation in Proposition 7 in which they have identical savings and are strictly rationed, which implies price-level determination.

It remains to prove point iii). We proceed in two steps.

Step 1: Construction of the equilibrium in which only the official market is active at date 1. The equilibrium is such that agents with a date-0 good endowment below a threshold y (possibly all of them) invest their entire endowment in money and the others (if any) invest up to y. We construct y as follows. Let $y = \min\{\max_{i \in \mathcal{I}} e_i, y'\}$, where y' solves

$$-\delta_{G,1} \frac{\sum_{i \in \mathcal{I}} \min(e_i, y') - y'}{(\sum_{i \in \mathcal{I}} \min(e_i, y'))^2} = \rho, \tag{43}$$

or is equal to $+\infty$ if this equation admits no solution. The right-hand side of the condition defining case iii) ensures that y' is larger than $\min_{i \in \mathcal{I}} e_i$. Agent $i \in \mathcal{I}$ invests $x_i = \min\{e_i, y\}$. The left-hand side of the condition defining case iii) implies that agents are strictly rationed at date 1 with heterogeneous holdings and thus the case of financial repression in Proposition 7 applies.

Step 2: Introduction of date-1 unofficial trading. We know from the proofs of Lemma 5 and Proposition 7 that one can construct equilibria with an unofficial market with sufficiently small unofficial trading volume that the payoffs can be made arbitrarily close to that in the absence of such an unofficial market. So, in the case in which y defined above is larger than $\max_{i \in \mathcal{I}} e_i$, it is possible to construct such an equilibrium in which all agents still find it optimal to invest their entire endowment in money at date 0 at P_0^* and there are two active price levels at date 1, as in the equilibria constructed in the proof of Lemma 5. We omit for brevity the case in which some agents cap their date-0 investment. The only additional complication in this case is that the threshold y now depends on the anticipated size of the date-1 unofficial trading volume. □

Proposition 13 shows that the date-0 determination of the price level may or may not come with that of the date-1 price level. Even if the date-1 price level is not determined, by trading at a fixed price at date 0, the state ensures that the multiple date-1 equilibria translate into the indeterminacy of quantities, not prices at date 0.

More precisely, the expectation of financial repression and the corresponding rationing at date 1 does constrain date-0 portfolios. This happens in case iii) when $-\delta_{G,1}[1 - e_i/(ne)]/e < \rho$ for some $i \in \mathcal{I}$. In this case, more affluent agents will optimally cap their holdings of money as rationing limits the return they obtain on money. To the extent that the cap does not prevent heterogenous portfolios, the resulting heterogeneity of money holdings leads to the possibility of unofficial trades as in Proposition 7. When $-\delta_{G,1}$ is small enough, as in case ii), all agents cap their holdings of money leading to homogenous portfolios, which ensures date-1 price level determination.

7 Conclusion

This paper studies the extent to which distinctive capacities of the state—issuing money, declaring taxes, and implementing bankruptcy, together with its trades of money for other goods, imply that public financial policy determines the price level. Our concept of price-level determination is robust in the sense that we set all agents free to trade whichever quantities at whichever prices they want. In addition to characterizing policies that do determine the price level, we also offer a description of the set of predictable price levels even when this is not a singleton. We obtain realistic predictable outcomes such as the rise of unofficial prices and that of endogenous intermediaries in this case of price-level indetermination. Our strategically closed framework also unveils the important out-of-equilibrium dimensions of policies that shape the equilibrium outcomes behind the curtains in Walrasian environments.

Our focus has been on economies in which neither money nor other public liabilities play an important role at overcoming frictions. A natural route for future research is to incorporate such a role in the analysis. This would in particular allow us to develop a normative analysis, assessing for example the welfare costs of price-level indetermination. Other situations that our strategically closed model is well-suited to study are that of the coexistence of multiple (private or/and public) monies. We also leave this for future work.

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